The Road to Two Theorems of Logic

William Craig
University of California, Berkeley

Abstract

Work on how to axiomatize the subtheories of first-order theory in which only a proper subset of their extra-logical vocabulary is being used led to a theorem on recursive axiomatizability and to an interpolation theorem for first-order logic. There were some fortuitous events and several logicians played a helpful role.

Keywords: recursive axiomatizability, interpolation for first-order logic, Robinson's joint consistency theorem, Beth's definability theorem, linear reasoning and the Herbrand-Gentzen theorem.

The preceding article ends with a brief description of the main result of [Cr 60], an axiomatization result regarding first-order logic with equality, that is related to elimination problems. In the present article, I want to describe how from work on that result I was led to two further theorems of logic, each related to it in a different way. One of these is an observation about recursive axiomatizability (cf. [Cr 1953]) and [Cr 1956]). The other is an interpolation theorem for first order logic with equality (Lemma 3 of [Cr 1957(b)], which is derived there from Theorem 5 of [Cr 1957(a)], an interpolation theorem for first-order logic without axioms for equality.) As will be seen, some events turned out to be fortuitous. Also, several logicians played a helpful role.

The axiomatization result of [Cr 60] was originally presented, in a somewhat different way, in Chapter IV of my Harvard Ph.D. thesis in philosophy [Cr 1951]. In the *Abstract* preceding the thesis, its main result is summarized as is shown below. For suggestiveness, the places of B and C in the original version have been interchanged below. Also, $\rightarrow$ is replaced below by $\supset$. Further, $FC1=$ abbreviates: First-order functional calculus with identity.

Theorem 1 *If A and C are any formulas of $FC1=$ such that the conditional $A \supset C$ is a theorem of $FC1=$, then there exists an 'intermediate' formula B of $FC1=$ which satisfies the following four conditions:*

(1) *$A \supset B$ is a theorem of $FC1=$.*

(2) *$B \supset C$ is a theorem of $FC1=$.*

(3) *With the possible exception of the identity sign, a predicate occurs in B only if it occurs both in A and in C.*

(4) *A certain fairly close relationship exists between the 'structure' of A and that of B. (The relationship cannot be described here.)*

Although I was aware of the mathematical interest of questions related to elimination problems in logic, my main aim, initially unfocused, was to try to use methods and results from logic to clarify or illuminate a topic that seems central to empiricist programs: In epistemology, the relationship between the external world and sense data; in philosophy of science, that between theoretical constructs and observed data.

To bring out how, in philosophy of science, the above *Theorem* may illuminate this relationship, or at least raise or sharpen questions about it, it will help to make the following two assumptions, each generally unrealistic:

(i) The part of physics or biology under consideration has been, or can be, axiomatized by a set of first-order sentences.

(ii) The set of extra-logical predicates that occur in the axioms can be partitioned into two disjoint sets, so that those in one set refer to theoretical constructs and those in the other to observed data.

To simplify exposition, it will also be assumed, temporarily, that the axiom set that is given is finite. Let A be a conjunction of the formulas in this set. Let R and S be the set of those predicates occurring in A that are *theoretical* or *observational*, respectively, in the sense just indicated. Let a formula C of $FC1=$ be *observational*, or an S*-formula*, if and only if every extralogical predicate that occurs in it belongs to S. Then C is an observational consequence of A if and only if C is an S-formula and $A \supset C$ is a theorem of $FC1=$. Now let A^∇ be the set of those B such that for some observational consequence C of A, B is an 'intermediate' formula in the sense of the above *Theorem.* By condition (3), any B in A^∇ is an S-formula, while, by condition (1), it is a consequence of A. Hence any B in A^∇ is an observational consequence of A. From the definition of A^∇ and from condition (2) there follows that, for every observational consequence C of A, there is some B in A^∇ such that C is a consequence of B. Thus, conditions (1), (2), and (3) together imply that A^∇ is an axiomatization of the set of observational consequences C of A.

The aim of imposing the additional condition (4) was to make A^∇ an axiomatization that could turn out to be useful or informative. Evidently, the usefulness of A^∇ largely depends on how close a relationship can be established between the 'structure' of A and those of the 'intermediate' formulas B that together constitute A^∇. Thus, most of the effort in proving the *Theorem* was directed toward establishing as close a relationship as seemed possible, thereby obtaining a strong condition (4).

In retrospect, it seems ironic that apparently nowhere in [Cr 51] are there signs of an awareness that, even without condition (4), the *Theorem* might be of interest. It should perhaps be added that the value of A^∇ as an axiomatization does not seem to be diminished if one replaces (3) by the following weaker condition.

> $(3)^-$ With the possible exception of the identity sign, a predicate occurs in B only if it occurs in C.

Apparenty, (3) was used, rather than $(3)^-$, in formulating the *Theorem* not because there was a felt need for it, but because the proof that was used yielded (3) quite naturally, initially for the quantifier-free case. Also esthetic reasons may have played a role, since (3) is in a certain sense symmetric, favoring neither A nor C.

The assumption that the axiom set that is given is finite can be relaxed. It suffices to assume that the set of Gödel numbers of the formulas in it is recursive and that the set of Gödel numbers of the predicates in S also is recursive. This can be shown by using, in addition to the *Theorem*, the compactness and completeness of first-order logic. (Cf. [Cr 60] or Chapter IV of [Cr 51].)

Work on [Cr 51] took up three years. Supported by a generous *Howison travelling fellowship in philosophy* from the University of California, Berkeley, and also, because of service in the U.S. Army, by the enlightened *G.I. bill*, I was able to spend the first of these at the E.T.H. in Zürich, and the second at Princeton University, auditing courses and seminars at each institution. Professors Bernays, Church, and Quine made valuable suggestions during the first, second, or third year respectively. Probably most important were talks with Professor Bernays who, during walks in well-kept parks bordering the shores of Lake Zürich, gently but firmly expressed scepticism about some of the philosophical views that I then held and also directed me to the writings of Herbrand and of Gentzen.

In 1950, at the International Congress of Mathematics held in Cambridge, Massachusetts, I gave a short contributed talk, unrelated to [Cr 51], on a consequence of Turing's paper [Tu]. Kleene and Tarski, neither of whom I had met before, were in the audience. After the talk, aiming to encourage a young logician, Tarski came up to introduce himself and also to introduce me to Kleene. Our talk turned to the thesis on which I was then working. It turned out that Kleene had obtained results, later published in [Kl], such that his results and those of my thesis complement each other.

Let $\{C_i(S) : i < \omega\}$ be a set of first-order formulas. Assume that the set S of predicates that occur in some $C_i(S)$ is finite. Also assume that $\{C_i(S) : i < \omega\}$ is closed under the relation of logical consequence and admits no models that are finite. Further assume that the set of Gödel numbers of the formulas in $\{C_i(S) : i < \omega\}$ is recursively enumerable. Then, according to [Kl] (see also [CV]), one can find a single formula $A(R, S)$, in which, besides the predicates in S, there occur predicates in an additional set R, such that $\{C_i(S) : i < \omega\}$ is the set of S-consequences of $A(R, S)$. Then, according to [Cr 51], the set $\{C_i(S) : i < \omega\}$ can be axiomatized by the set A^∇, whose set of Gödel numbers is recursive. Thus, the joint effect of [Kl] and [Cr 51] is that for many sets $\{C_i(S) : i < \omega\}$ of first-order formulas, recursive enumerability implies axiomatizability by a recursive subset.

Several months after I had noticed this effect, I observed that a similar conclusion holds for any formal system L that satisfies certain very mild conditions. The argument is very simple and depends neither on [Kl] nor on [Cr 51]. For one set of conditions it can be given as follows. (Cf. [Cr 53], which received very useful input from Kleene and in which there are also mentioned other sets of conditions that can be used instead.)

Let L be any formal system. Let $F \vdash_L F'$ if and only if F and F' are formulas of L and there is a derivation in L from F of F'. Assume that L has a symbol $\wedge$ for conjunction and that for any formula E of L there is a unique n, $0 \leq n < \omega$, and a unique formula F such that E is the $(n+1)$-fold conjunction $F^{\hat{n}}$ each of whose $(n+1)$ conjuncts is F. Also assume that, for any formula F, and for any n, $0 \leq n < \omega$, both $F \vdash_L F^{\hat{n}}$ and $F^{\hat{n}} \vdash_L F$. Furthermore, assume that there has been given an effective Gödel numbering of the formulas of L, i.e., a one-one correspondence between the set $\{m : 0 \leq m < \omega\}$ of natural numbers and the set of formulas of L such that, if G_m is the formula that corresponds to m, then, given m, one can find G_m and, given G_m, one can find m.

Consider any set $\{G_{h(n)} : n < \omega\}$ of formulas that is closed under $\vdash_L$, such that h is a recursive function. Since the set is closed under $\vdash_L$, therefore it includes as subset the set $\{G^{\hat{n}}_{h(n)} : n < \omega\}$. Now consider the closure under $\vdash_L$ of this subset. Since it is closed under $\vdash_L$, therefore this closure contains as member any formula $G_{h(n)}$, such that $G^{\hat{n}}_{h(n)}$ is in the subset. Since $\{G_{h(n)} : n < \omega\}$ is closed under $\vdash_L$, there now follows that it equals the closure under $\vdash_L$ of its subset $\{G^{\hat{n}}_{h(n)} : n < \omega\}$.

To see that membership in this subset is decidable, let any formula E of L be given. One can find the unique formula F and the unique n, $0 \leq n < \omega$, such that $E = F^{\hat{n}}$. Then E is in $\{G^{\hat{n}}_{h(n)} : n < \omega\}$ if and only if $F = G_{h(n)}$.

A first reaction to this axiomatizability result tends to be one of surprise and perhaps disappointment. While considerable effort had gone into proving the axiomatizability result of [Cr 51], the above argument shows that recursive axiomatization can often be achieved without effort. This is why this axiomatization method has sometimes been called *Craig's trick.* My friend Robin Gandy called it *Craig's swindle.*

The article [Cr 53] describing this method of axiomatization is fairly technical and philosophical issues were mentioned only briefly. Encouraged by Professor Hempel, I described the same method in a second article, intended for philosophers ([Cr 58]). It is longer, less technical, and devotes some space to philosophical issues.

For several years, I occupied myself with topics in logic that are unrelated to those of my thesis [Cr 51]. My interest returned to [Cr 51] when I began to realize that the *Theorem* cited at the beginning of this article is also of interest in the simpler form that results when one omits condition (4).

Interchange of ideas with Richard Büchi was one of the two causes of my renewed interest. I believe that I first met him at a summer meeting of the American Mathematical Society in Ann Arbor, which I attended, with the encouragement of my colleague Haskell Curry who at that time was my colleague in the Mathematics Department of the Pennsylvania State University and who, as on several other such occasions, had offered me a ride. Büchi made me aware of the importance of model-theoretic notions which at that time had begun to emerge. Once I became familiar with the notion *EC* of *elementary class* and the notion *PC* of *pseudo-elementary class*, it was simply a matter of "connecting the dots" to see that in the above simpler form, i.e., without condition (4), the *Theorem* is equivalent to the following separation theorem: For any two *PC* classes $\mathbf{K}$ and $\mathbf{K}'$ that are disjoint there is an *EC* class $\mathbf{J}$ that separates them, i.e., such that $\mathbf{K} \subseteq \mathbf{J}$ and $\mathbf{K}' \subseteq -\mathbf{J}$. (See §4 of [Cr 57(b)] and also [BC], which contains some further applications, due to Büchi.)

The second cause of my renewed interest was the appearance in [Be] of Beth's theorem. It states that, for first-order logic with equality, implicit definability implies explicit definability. Professor Church, who was then the editor of reviews for the *Journal of Symbolic Logic*, asked me to review Beth's paper. It was evident that a result of this kind would be of great importance. However, upon careful reading I noticed what seemed to me to be a gap in the proof. For several weeks I was unable to fill it and was uncertain whether the result claimed by [Be] was true. (In retrospect, I think that the proof can be mended by filling in some details.) Then there occurred to me a simple way of deriving Beth's theorem from the interpolation theorem for first-order logic with equality, i.e., from the above simpler form of the *Theorem* where condition (4) is omitted.

Without further delay, I wrote a review in which I mentioned what seemed to me to be the gap in the proof but also stated that Beth's theorem does hold and, in fact, follows from an interpolation result for first-order logic. I stated the result briefly, but gave no indication of its proof. Church returned the review, pointing out that it is against the policy of the *Journal* to allow inclusion or mention in reviews of unpublished results. I then revised the review accordingly and also set out to write up, for publication, proofs of two related interpolation theorems. For first-order logic without axioms for equality, this is Theorem 5 of [Cf 57(a)] and also Lemma 1 of [Cr 57(b)]. For first-order logic with axioms for equality, this is Lemma 3 of [Cr 57(b)], which is derived from Lemma 1. From Lemma 3 there is then derived Beth's Theorem and also a generalization of it from function symbols and predicates to terms and formulas. (Cf. Theorems 1 or 2, respectively, of [Cr 57(b)].)

It should be noted that Beth's Theorem can also be derived from Abraham Robinson's consistency theorem, which had recently appeared in [Ro]. In the case of first-order logic, the interpolation theorem and Robinson's consistency theorem are now known to be equivalent. (Cf. [Eb], Proposition 7.1.5.)

Herbrand's Theorem in [He] and Gentzen's *Extended Hauptsatz* in [Ge] deal with first-order logic without axioms for equality. They are essentially equivalent. Each is often used to reduce problems concerning formulas with quantifiers to related problems concerning formulas that are quantifier-free. Regarding quantifier-free formulas, Boole's treatment of *development* readily suggests how for any conditional $A \supset C$ that is quantifier-free and tautologous, such that the set S of predicates that occur both in A and in C is nonempty, one can obtain a quantifier-free B whose predicates all belong to S such that both $A \supset B$ and $B \supset C$ are tautologous. Thus, it is natural to try to use the Herbrand-Gentzen Theorem for "lifting" interpolation results from the quantifier-free level to quantificational levels.

In [Cr 51] this was done in a way that is not easy to follow. The argument used in [Cr 57(a)] also is lengthy, but the underlying plan is quite clear. The Herbrand-Gentzen Theorem, formulated there by means of a set H of rules, is also given there another formulation in terms of a system of *linear reasoning.* The system consists of a set of *L-rules*, each of which is a one-premiss rule such that, in each case, the conclusion is a logical consequence of the premiss. An *L-deduction* is defined to be any sequence $A_0, \ldots, A_n$ such that, for any m, $0 \leq m < n$, A_m yields A_{m+1} by the use of one of the L-rules.

Altogether there are eleven L-rules, ten of which come in five pairs, each pair consisting of an *a-rule* and a *b-rule* that, in a certain sense, are dual to each other. The remaining L-rule is the following one of *Matrix Change*:

> If (P) is any prefix and if $M \supset M'$ is quantifier-free and tautologous, then from $(P)M$ one may infer $(P)M'$.

Evidently, there are uses of Matrix Change where the predicates that occur in $(P)M$ and those that occur in $(P)M'$ do not form the same set. In contrast, each of the other L-rules is such that the predicates that occur in the premiss and those that occur in the conclusion form the same set.

An L-deduction $A_1, \ldots, A_n$ shall be an *a–M–b deduction* if and only if there are k and m, $0 \leq k < m \leq n$, such that in the initial phase $A_0, \ldots, A_k$, if nonempty, one only uses a-rules, in the middle phase $A_{k-1}, \ldots, A_m$ one only uses Matrix Change, and in the end phase $A_{m+1}, \ldots, A_n$, if nonempty, one only uses b-rules. Thus, in an a–M–b deduction, any change of predicates must occur in the middle phase.

In [Cr 57(a)] it is shown that the Herbrand-Gentzen Theorem can be given the following new formulation:

> If A is any conjunction of formulas in prenex normal form and C any disjunction of such formulas and if $A \supset C$ is valid, there then is an $a-M-b$ deduction $A_0, \ldots, A_n$ such that $A_0 = A$ and $A_n = C$. Moreover, the uses of Matrix Change in the middle phase can be replaced by a single use of Matrix Change.

Assume that $A \supset C$ is valid and that the set S of predicates that occur both in A and in C is nonempty. Since every formula is logically equivalent to one in prenex normal form and with the same set of predicates, one can assume without loss of generality that A is a conjunction of formulas in prenex normal form and C a disjunction of such formulas.

Consider any a–M–b deduction $A_0, \ldots, A_m, A_{m+1}, \ldots, A_n$ such that $A_0 = A$, $A_n = C$ and the only Matrix Change is the step from $A_m = (P)M$ to $A_{m+1} = (P)M''$. Since $M \supset M''$

is tautologous and since $S \neq \emptyset$ therefore there is some quantifier-free M' whose predicates all belong to S such that both $M \supset M'$ and $M' \supset M''$ are tautologous. There follows that both $(P)M \supset (P)M'$ and $(P)M' \supset (P)M''$ are valid. Hence also both $A \supset (P)M'$ and $(P)M' \supset C$ are valid. Thus, since all the predicates that occur in $(P)M'$ belong to S, therefore $(P)M'$ is a desired "intermediate" formula.

Let $A_0, \ldots, A_m$ be the initial phase of the above a–M–b deduction $A_0, \ldots, A_m, A_{m+1}, \ldots, A_n$, so that $A_0 = A$ and $A_m = (P)M$. Since any L-rule being used in this phase is an a-rule and since certain structural features of a formula are preserved by the use of an a-rule, therefore the structure of A and that of $(P)M$ are in certain respects related. Since $(P)M$ yields $(P)N$ by Matrix Change, therefore their respective structures also are related, although in different respects. Although there are some ways in which the relationship can be tightened, especially, as is shown in [Cr 60], in the case of first-order logic with equality, it seems, in retrospect, to be an exaggeration to describe, as was done in the Abstract of [Cr 51], the resulting relationship of the structure of $(P)N$ to that of A as 'fairly close'. Nevertheless, it is close enough so that, given any conjunction A of formulas in prenex normal form and any $(Q)N'$ in prenex normal form, one can decide whether or not there is some $(Q)M'$ such that $(Q)M'$ yields $(Q)N'$ by Matrix Change and A yields $(Q)M'$ by the use of a-rules.

The interpolation theorem for first-order logic with axioms for equality can be derived from the interpolation theorem just described by arguments that are well known (see, for example, the proof of Lemma 3 of [Cr 57(b)]).

William Craig
Department of Philosophy
University of California
Berkeley, CA 94720–2390

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