Lecture 1

Monday, January 13, 2020

Course info. What is logic?

PHIL 279 Lec 01: Logic I

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Department of Philosophy
University of Calgary

Winter Term 2020

UNIVERSITY OF CALGARY
FACULTY OF ARTS
Department of Philosophy

Course Info

Team

Instructor	**Dr. Richard Zach** (he/him) Social Sciences 1230 MW 12:00–12:50
TAs	**Amir Kiani** (he/him) Social Sciences 1231 T 11:00–12:00, R 11:00–12:00 **Brent Odland** (he/him) Social Sciences 1242 R 12:00–13:00
PASS leader	**Sarah Hatcher** (she/her)

Textbook

forall$x$
CALGARY

An Introduction to
Formal Logic

Available on D2L, and at
forallx.openlogicproject.org

Assessment

- 5 problem sets (50%)
  - Some written
  - Some online using carnap.io
- 3 tests (45%)
  - In class
  - Closed book
- 10 quizzes (5%)
  - Online on D2L
  - Open book

Course components

- Lectures
- Tutorials
  - R 8:30–9:20, Education C 284 (Tut 1—Brent)
  - R 9:00–9:50, Science A 107 (Tut 3—Amir)
  - R 10:00–10:50, Science Theatres 59 (Tut 2—Amir)
- PASS Sessions
  - Tutorials start this week
  - PASS sessions start next week
- D2L website discussion forum

House rules

- Be civil and behave like adults: no sexist, racist, etc. jokes
- Don’t distract others
  - Phones to silent
  - Avoid use of devices for anything not related to class
- Collaborate and study together, but turn in only your own independent work
- Don’t give away answers
- Don’t cheat on quizzes and exams

Read the course outline!

- Official outline covers all policy questions
- Outline is binding agreement and you are responsible for knowing policies
- Available on D2L and at phil.ucalgary.ca/courses

Questions?

Stay safe and well

Download UC Emergency Mobile app — The app delivers alerts and updates about serious incidents that could affect you.

Know who to contact

- **911:** critical life-safety emergencies
- **Campus Security:** non-urgent safety or security concerns **403.220.5333**
- **Student Wellness Services:** medical and mental health services and after-hours telephone support for mental health concerns **403.210.9355**
- **Sexual Violence Support Advocate:** assistance if you've experienced sexual violence **403.220.2208 | svsa@ucalgary.ca**

Safewalk — Volunteers will walk you anywhere on campus 24/7 **403.220.5333**

RUN HIDE FIGHT

Run, Hide, Fight — In the unlikely event of an active assailant, escape if you can, hide if you can't, and fight only if you have to.

UNIVERSITY OF CALGARY

Arguments and validity
Chapters 1–3

An easy puzzle

Where is Sanjiv from?

Sanjiv is from Calgary or from Edmonton.

Sanjiv is not from Edmonton.

Arguments and sentences

Argument 1

Sanjiv is from Calgary or from Edmonton.
Sanjiv is not from Edmonton.
Therefore, Sanjiv is from Calgary.

- "Therefore" indicates that the last sentence (supposedly) **follows from** the first two.
- The last sentence is called the **conclusion**.
- The others are called the **premises**.

Valid and invalid arguments

Argument 2

Mandy enjoys skiing or hiking.
Mandy doesn't enjoy hiking.
∴ Mandy enjoys skiing.

Argument 3

Mandy enjoys skiing or hiking.
Mandy enjoys skiing.
∴ Mandy doesn't enjoy hiking.

Valid and invalid arguments

Argument 2

Mandy enjoys skiing or hiking.
Mandy doesn't enjoy hiking.
∴ Mandy enjoys skiing.

Argument 3

Mandy enjoys skiing or hiking.
Mandy enjoys skiing.
∴ Mandy doesn't enjoy hiking.

- Argument 2 is **valid**: whenever the premises are true, the conclusion is true.

Valid and invalid arguments

Argument 2

Mandy enjoys skiing or hiking.
Mandy doesn't enjoy hiking.
∴ Mandy enjoys skiing.

Argument 3

Mandy enjoys skiing or hiking.
Mandy enjoys skiing.
∴ Mandy doesn't enjoy hiking.

- Argument 2 is **valid**: whenever the premises are true, the conclusion is true.
- Argument 3 is **invalid**: there is a case where the premises are true and the conclusion isn't (Mandy enjoys both skiing and hiking).

Validity

Definition

A **case** is some hypothetical scenario that makes each sentence in an argument either true or false.

Definition

An argument is **valid** if there is no case where all its premises are true and the conclusion is false.

Definition

An argument is **invalid** if there is at least one case where all its premises are true and the conclusion is false (i.e., if it is no t valid).

A harder puzzle

Valid or invalid?

Sara is from Calgary or Edmonton.

Mariusz is from Calgary unless he enjoys hiking.

If Mariusz is from Calgary, Sara isn't.

neither Sara nor Mariusz enjoy hiking.

$\therefore$ Sara is from Edmonton.

Validity in virtue of form

Argument 1

Sanjiv is from Calgary or from Edmonton.
Sanjiv is not from Edmonton.
∴ Sanjiv is from Calgary.

Argument 2

Mandy enjoys skiing or hiking.
Mandy doesn't enjoy hiking.
∴ Mandy enjoys skiing.

Validity in virtue of form

Argument 1

Sanjiv is from Calgary or from Edmonton.
Sanjiv is not from Edmonton.
$\therefore$ Sanjiv is from Calgary.

Argument 2

Mandy enjoys skiing or hiking.
Mandy doesn't enjoy hiking.
$\therefore$ Mandy enjoys skiing.

Form of arguments 1 & 2

X or Y.
Not Y.
$\therefore X$.

Some valid argument forms

Disjunctive syllogism

X or Y.

Not Y.

$\therefore$ X.

Modus ponens

If X then Y

X

$\therefore$ Y

Hypothetical syllogism

If X then Y

If Y then Z

$\therefore$ If X then Z

Symbolizing arguments

Symbolization key

S: Mandy enjoys skiing

H: Mandy enjoys hiking

Argument 2

Mandy enjoys skiing or Mandy enjoys hiking.	$(S \lor H)$
Not: Mandy enjoy hiking.	$\neg H$
$\therefore$ Mandy enjoys skiing.	$\therefore S$

The language of TFL

- **Sentence letters**, such as 'H' and 'S', to symbolize basic sentences ('Mandy likes hiking')
- **Connectives**, to indicate how basic sentences are connected

  $\vee$ either ... or ...

  $\wedge$ both ... and ...

  $\rightarrow$ if ... then ...

  $\neg$ not ...

  This can get complicated, e.g.:

  "Mandy enjoys skiing or hiking, and if she is from Edmonton, she doesn't enjoy both."

$$((S \vee H) \wedge (E \rightarrow \neg(S \wedge H)))$$

What is logic?

What is logic?

- **Logic is the science of what follows from what.**
  - Mandy is from Calgary.
    Everyone from Calgary likes hiking.
    $\therefore$ Mandy likes hiking.
  - Mandy is from Calgary.
    Everyone who likes hiking is from Calgary.
    $\therefore$ Mandy likes hiking.
- Logic investigates what makes the first argument **valid** and the second **invalid**.

What is formal logic?

- Studies logical properties of formal languages (not English)
  - Logical consequence (what follows from what?)
  - Logical consistency (when do sentences contradict one another?)
- Expressive power (what can be expressed in a given formal language, and how?)
- Formal models (mathematical structures described by formal language)
- Inference and proof systems (how can it be proved that something follows from something else?)
- (Meta-logical properties of logical systems)

Plan for the course

- Truth-functional logic (TFL)
  - Symbolization in the formal language of TFL ($H, \vee, \wedge, \rightarrow, \neg$)
  - Testing for validity: truth-tables
  - Proofs in natural deduction
- First-order logic (FOL)
  - More fine-grained symbolization ($H(a, b)$, $\forall$ 'every', $\exists$ 'some', $=$)
  - Semantics: interpretations
  - Proofs in natural deduction
- Some advanced topics: expressive adequacy, connections between truth tables and proofs

What is logic good for? (Philosophy)

- Logic originates in philosophy (Aristotle), traditionally considered a sub-discipline of philosophy
- Valid argument cornerstone of philosophical research
- Formal tools of logic useful to make intuitive philosophical notions precise
  - Possibility and necessity
  - Time
  - Moral obligation and permissibility
  - Belief and knowledge
- Applies to semantics of natural language (philosophy of language, linguistics)

What is logic good for? (Mathematics)

- Formal logic developed in the quest for foundations of mathematics (19th C.)
- Logical systems provide precise foundational framework for mathematics
  - Axiomatic systems (e.g, geometry)
  - Algebraic structures (e.g., groups)
  - Set theory (e.g, Zermelo–Fraenkel with Choice)
- Precision
  - Formal language makes claims more precise
  - Formal structures can point to alternatives, unveil gaps in proofs
  - Formal proof systems make proofs rigorous
  - Formal proofs make mechanical **proof checking** and **proof search** possible

What is logic good for? (Computer Science)

- Combinational logic circuits
- Database query languages
- Logic programming
- Knowledge representation
- Automated reasoning
- Formal specification and verification (of programs, of hardware designs)
- Theoretical computer science (theory of computational complexity, semantics of programming languages)

Lecture 2

Wednesday, January 15, 2020

Introduction to symbolization

Symbolization in TFL
Chapters 4–6

Symbolizing arguments

Recall:

Argument 2

Mandy enjoys skiing or Mandy enjoys hiking.
Not: Mandy enjoy hiking.
$\therefore$ Mandy enjoys skiing.

Form of argument 2

S or H.
Not H.
$\therefore S$.

Symbolization of argument 2 in TFL

$(S \vee H)$
$\neg H$
$\therefore S$

Symbolization keys

Definition

A symbolization key is a list that pairs **sentence letters** with the basic English sentences they represent.

For instance:

Symbolization key

S: Mandy enjoys skiing
H: Mandy enjoys hiking

Symbolization keys

- Sentence letters are uppercase letters, possibly with subscripts (e.g., H_1, H_2)
- Usually the symbolization key is given to you
- It should not be possible to break down the "basic sentences" represented by sentence letters.
  For instance:

  A: Mandy likes skiing or hiking

  is bad.

Paraphrase

- Successful symbolization sometimes requires **paraphrase** to ensure basic sentences appear explicitly
- Two things to watch for: pronouns, and coordination
- Pronouns stand in for singular terms (e.g., names): replace pronouns by those
- “or”, “and”, “both”, “neither” can connect sentences but also noun phrases and verb phrases: paraphrase them so they connect complete clauses

Pronouns

Example

If Mandy enjoys hiking, she also enjoys skiing.

Replace “she” by “Mandy”:
If [Mandy enjoys hiking], [Mandy enjoys skiing].

Coordination of noun phrases

Example

Mandy and Sanjiv enjoy hiking.

[Mandy enjoys hiking] and [Sanjiv enjoys hiking].

Example

Sanjiv is from **Edmonton or Calgary**.

[Sanjiv is from Edmonton] or [Sanjiv is from Calgary].

Exercise caution!

Good

Mandy and Sanjiv ate pizza.

[Mandy ate pizza] and [Sanjiv ate pizza].

Bad

Mandy and Sanjiv ate the whole pizza.

[Mandy ate the whole pizza] and [Sanjiv ate the whole pizza].

Coordination of verb phrases

Example

Mandy enjoys **skiing or hiking**.

[Mandy enjoys skiing] or [Mandy enjoys hiking].

Example

If Sanjiv enjoys **skiing and hiking**, he is from Calgary.

If [Sanjiv enjoys skiing] and [Sanjiv enjoys hiking], then [Sanjiv is from Calgary].

Negation

Paraphrase grammatical negation ("is not", "does not") using the corresponding basic sentence prefixed by "**it is not the case that**"

Symbolize "it is not the case that A" as $\neg A$.

Example

Mandy doesn't enjoy skiing.

It is not the case that [Mandy enjoys skiing].

It is not the case that S.

$\neg S$

Conjunction

Paraphrase sentences containing “and”, “but”, “even though”, and “although” using “**both A and B**”

Symbolize “both A and B” as $(A \wedge B)$.

Example

Even though Mandy is from Edmonton, she enjoys hiking.

Both [Mandy is from Edmonton] and [Mandy enjoys hiking].

Both E and H.

$(E \wedge H)$

disjunction

Paraphrase sentences containing "or" using "**either** A **or** B"

Symbolize "either A or B" as $(A \vee B)$.

Example

Sanjiv is from Calgary or from Edmonton.

either [Sanjiv is from Calgary] or [Sanjiv is from Edmonton].

either C or E.

$(C \vee E)$

Ignore the suggestion that "either . . . or . . . " is exclusive. We'll always treat it as inclusive unless explicitly stated.

Conditional

Paraphrase sentences of the forms: "if A, B"
"A only if B"
"B if A" (note order is reversed!)
"B provided A"
using "**if A then B**"

Symbolize "if A then B" as $(A \to B)$.

Example

Mandy enjoys hiking if Sanjiv is from Calgary.

If [Sanjiv is from Calgary] then [Mandy enjoys hiking].

If C then H.

$(C \to H)$

Mix & match

Example

Mandy doesn't enjoy hiking, provided Sanjiv is from Calgary or Edmonton.

If [Sanjiv is from Calgary or Edmonton] then [Mandy doesn't enjoy hiking].

If [either [Sanjiv is from Calgary] or [Sanjiv is from Edmonton]] then [it is not the case that [Mandy enjoys hiking]].

If [either C or E] then [it is not the case that H].

$((C \vee E) \rightarrow \neg H)$

If and only if

Example

Mandy enjoys hiking if and only if she enjoys skiing.

H if and only if S.

$(H \leftrightarrow S)$

Lecture 3

Friday, January 17, 2020

TFL and truth tables for $\neg$, $\wedge$, $\vee$

TFL and truth tables
Chapters 8–10

Sentence letters and connectives

- Symbolization involves **sentence letters** like H and **connectives** ($\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$)
- Recall that a **case** makes basic sentences **true** or **false** (and never both)
- So if we can determine **truth conditions** for sentences involving connectives, we can assign **true** and **false** also to results of symbolization.

Example

$(H \wedge S)$ is true if and only if H is true and S is also true.

Suppose a case makes H true and S false.

In that case, $(H \wedge S)$ would be ???.

Sentence letters and connectives

- Symbolization involves **sentence letters** like H and **connectives** ($\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$)
- Recall that a **case** makes basic sentences **true** or **false** (and never both)
- So if we can determine **truth conditions** for sentences involving connectives, we can assign **true** and **false** also to results of symbolization.

Example

$(H \wedge S)$ is true if and only if H is true and S is also true.

Suppose a case makes H true and S false.

In that case, $(H \wedge S)$ would be **false**.

Conjunction

Definition

$(\mathscr{A} \wedge \mathscr{B})$ is true iff $\mathscr{A}$ is true and $\mathscr{B}$ is true, and false otherwise.

Characteristic truth table:

$\mathscr{A}$	$\mathscr{B}$	$(\mathscr{A} \wedge \mathscr{B})$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

Definition

$(\mathscr{A} \vee \mathscr{B})$ is true iff $\mathscr{A}$ is true or $\mathscr{B}$ is true (or both), and false otherwise.

Characteristic truth table:

$\mathscr{A}$	$\mathscr{B}$	$(\mathscr{A} \vee \mathscr{B})$
T	T	T
T	F	T
F	T	T
F	F	F

Negation

Definition

$\neg\mathscr{A}$ is true iff $\mathscr{A}$ is false.

Characteristic truth table:

$\mathscr{A}$	$\neg\mathscr{A}$
T	F
F	T

Sentences of TFL

Definition

1. Every sentence letter is a sentence.
2. If $\mathscr{A}$ is a sentence, then $\neg\mathscr{A}$ is a sentence.
3. If $\mathscr{A}$ and $\mathscr{B}$ are sentences, then
   - $(\mathscr{A} \wedge \mathscr{B})$ is a sentence.
   - $(\mathscr{A} \vee \mathscr{B})$ is a sentence.
   - $(\mathscr{A} \rightarrow \mathscr{B})$ is a sentence.
   - $(\mathscr{A} \leftrightarrow \mathscr{B})$ is a sentence.
4. Nothing else is a sentence.

The indicated connective is called the **main connective**.

Construction of sentences

- H is a sentence.
- S is a sentence.
- $(H \vee S)$ is a sentence.
- $(H \wedge S)$ is a sentence.
- $\neg(H \wedge S)$ is a sentence.
- $((H \vee S) \wedge \neg(H \wedge S))$ is a sentence.

(Main connective is **highlighted**.)

Examples of non–sentences

- $HikesMandy$
  single sentence letters
- $(H \neg S)$
  $\neg$ can't go between sentences
- $(H \wedge S \wedge C)$
  $\wedge$ combines only two sentences
- $(\neg H)$
  no parentheses around $\neg H$
- $(H \rightarrow (S \wedge C)$
  missing closing parenthesis
- $H \vee S$
  missing parentheses
- $[H \rightarrow (S \wedge C)]$
  only one kind of parentheses

Valuations

Definition

A **valuation** is an assignment of **T** or **F** to each sentence letter in a sentence or sentences.

Definition

The **truth value of a sentence** $\mathcal{S}$ on a valuation is:

1. if $\mathcal{S}$ is a sentence letter: the truth value assigned to it
2. if $\mathcal{S}$ is $\neg\mathcal{A}$: opposite of the truth value of $\mathcal{A}$
3. if $\mathcal{S}$ is $(\mathcal{A} * \mathcal{B})$: result of characteristic truth table of $*$ for truth values of $\mathcal{A}$ and $\mathcal{B}$.

Computing truth values

Valuation: H is **T**, S is **F**.

On this valuation:

- H is **T**.
- S is **F**.
- $(H \vee S)$ is **T** (because **T** $\vee$ **F** gives **T**).
- $(H \wedge S)$ is **F** (because **T** $\wedge$ **F** gives **F**).
- $\neg(H \wedge S)$ is **T** (because $\neg$**F** is **T**).
- $((H \vee S) \wedge \neg(H \wedge S))$ is **T** (because **T** $\wedge$ **T** gives **T**).

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\wedge$	$\neg$	$(H$	$\wedge$	$S))$
T	**F**								

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\wedge$	$\neg$	$(H$	$\wedge$	$S))$
T	**F**	**T**		**F**			**T**		**F**

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\wedge$	$\neg$	$(H$	$\wedge$	$S))$
T	**F**	**T**		**F**			**T**		**F**
			$\uparrow$						

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\wedge$	$\neg$	$(H$	$\wedge$	$S))$
T	**F**	**T**	**T**	**F**			**T**		**F**
			$\uparrow$						

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\wedge$	$\neg$	$(H$	$\wedge$	$S))$
T	**F**	**T**	**T**	**F**			**T**		**F**
								$\uparrow$	

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\wedge$	$\neg$	$(H$	$\wedge$	$S))$
T	**F**	**T**	**T**	**F**			**T**	**F**	**F**
								$\uparrow$	

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\wedge$	$\neg$	$(H$	$\wedge$	$S))$
T	**F**	T	T	F			T	**F**	F
						$\uparrow$			

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\wedge$	$\neg$	$(H$	$\wedge$	$S))$
T	**F**	T	T	F		**T**	T	**F**	F
						$\uparrow$			

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\wedge$	$\neg$	$(H$	$\wedge$	$S))$
T	**F**	T	**T**	F		**T**	T	F	F
					$\uparrow$				

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\wedge$	$\neg$	$(H$	$\wedge$	$S))$
T	**F**	T	**T**	F	**T**	**T**	T	F	F
					$\uparrow$				

Validity in TFL
Chapter 11

Validity

- In English: an argument is **valid** if there is no **case** where all premises are true and conclusion is false.
- A **case** must make every **basic sentence** true or false (and not both).
- In TFL, **valuations** make every **sentence letter** true or false (and not both).
- Also: every valuation makes every **sentence** true or false (and not both), and we can compute the truth value.

Validity in TFL

Definition

An argument is **valid in TFL** if there is **no** valuation in which all premises are **T** and the conclusion is **F**.

An argument is **invalid in TFL** if there is **at least one** valuation in which all premises are **T** and the conclusion is **F**.

Disjunctive syllogism

$$
\begin{array}{rl}
 & H \lor S \\
 & \neg S \\
\therefore & H
\end{array}
$$

H	S	$(H$	$\lor$	$S)$	$\neg$	S	H

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$$
\begin{aligned}
& H \lor S \\
& \neg S \\
\therefore\ & H
\end{aligned}
$$

H	S	$(H \lor S)$	$\neg S$	H
T	**T**			
T	**F**			
F	**T**			
F	**F**			

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
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Disjunctive syllogism

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\begin{array}{rl}
 & H \vee S \\
 & \neg S \\
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\end{array}
$$

H	S	$(H$	$\vee$	$S)$	$\neg$	S	H
T	**T**	**T**		**T**			
T	**F**	**T**		**F**			
F	**T**	**F**		**T**			
F	**F**	**F**		**F**			

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
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Disjunctive syllogism

$$
\begin{array}{rl}
 & H \vee S \\
 & \neg S \\
\therefore & H
\end{array}
$$

H	S	$(H$	$\vee$	$S)$	$\neg$	S	H
T	**T**	**T**	**T**	**T**			
T	**F**	**T**	**T**	**F**			
F	**T**	**F**	**T**	**T**			
F	**F**	**F**	**F**	**F**			

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$$
\begin{aligned}
& H \vee S \\
& \neg S \\
\therefore\ & H
\end{aligned}
$$

H	S	$(H$	$\vee$	$S)$	$\neg$	S	H
T	**T**	T	T	T		**T**	
T	**F**	T	T	F		**F**	
F	**T**	F	T	T		**T**	
F	**F**	F	F	F		**F**	

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$$
\begin{aligned}
& H \vee S \\
& \neg S \\
\therefore\ & H
\end{aligned}
$$

H	S	$(H$	$\vee$	$S)$	$\neg$	S	H
T	**T**	T	T	T	**F**	**T**	
T	**F**	T	T	F	**T**	**F**	
F	**T**	F	T	T	**F**	**T**	
F	**F**	F	F	F	**T**	**F**	

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$$
\begin{array}{r}
H \lor S \\
\neg S \\
\therefore\ H
\end{array}
$$

H	S	$(H$	$\lor$	$S)$	$\neg$	S	H
T	**T**	T	T	T	F	T	**T**
T	**F**	T	T	F	T	F	**T**
F	**T**	F	T	T	F	T	**F**
F	**F**	F	F	F	T	F	**F**

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$$
\begin{array}{rl}
 & H \vee S \\
 & \neg S \\
\therefore & H
\end{array}
$$

H	S	$(H$	$\vee$	$S)$	$\neg$	S	H
T	**T**	T	**T**	T	**F**	T	**T**
T	**F**	T	**T**	F	**T**	F	**T**
F	**T**	F	**T**	T	**F**	T	**F**
F	**F**	F	**F**	F	**T**	F	**F**

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$$
\begin{array}{rl}
 & H \vee S \\
 & \neg S \\
\therefore & H
\end{array}
$$

H	S	$(H$	$\vee$	$S)$	$\neg$	S	H	
T	**T**	T	**T**	T	**F**	T	**T**	$\leftarrow$
T	**F**	T	**T**	F	**T**	F	**T**	
F	**T**	F	**T**	T	**F**	T	**F**	
F	**F**	F	**F**	F	**T**	F	**F**	

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$$
\begin{aligned}
& H \vee S \\
& \neg S \\
\therefore\ & H
\end{aligned}
$$

H	S	$(H$	$\vee$	$S)$	$\neg$	S	H	
T	**T**	T	**T**	T	**F**	T	**T**	✓
T	**F**	T	**T**	F	**T**	F	**T**	←
F	**T**	F	**T**	T	**F**	T	**F**	
F	**F**	F	**F**	F	**T**	F	**F**	

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

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\begin{array}{r}
H \vee S \\
\neg S \\
\therefore\ H
\end{array}
$$

H	S	$(H$	$\vee$	$S)$	$\neg$	S	H	
T	**T**	T	**T**	T	**F**	T	**T**	✓
T	**F**	T	**T**	F	**T**	F	**T**	✓
F	**T**	F	**T**	T	**F**	T	**F**	←
F	**F**	F	**F**	F	**T**	F	**F**	

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$$
\begin{aligned}
& H \vee S \\
& \neg S \\
\therefore\ & H
\end{aligned}
$$

H	S	$(H$	$\vee$	$S)$	$\neg$	S	H	
T	**T**	T	**T**	T	**F**	T	**T**	✓
T	**F**	T	**T**	F	**T**	F	**T**	✓
F	**T**	F	**T**	T	**F**	T	**F**	✓
F	**F**	F	**F**	F	**T**	F	**F**	←

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$$
\begin{array}{r}
H \vee S \\
\neg S \\
\therefore\ H
\end{array}
$$

H	S	$(H$	$\vee$	$S)$	$\neg$	S	H	
T	**T**	T	**T**	T	**F**	T	**T**	✓
T	**F**	T	**T**	F	**T**	F	**T**	✓
F	**T**	F	**T**	T	**F**	T	**F**	✓
F	**F**	F	**F**	F	**T**	F	**F**	✓

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

An invalid argument

$$
\begin{aligned}
& H \lor S \\
& H \\
\therefore\ & \neg S
\end{aligned}
$$

H	S	$(H$	$\lor$	$S)$	H	$\neg$	S

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

An invalid argument

$$
\begin{array}{l}
H \vee S \\
H \\
\therefore \neg S
\end{array}
$$

H	S	$(H \vee S)$	H	$\neg S$
T	**T**			
T	**F**			
F	**T**			
F	**F**			

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

An invalid argument

$$
\begin{array}{l}
H \lor S \\
H \\
\therefore \neg S
\end{array}
$$

H	S	$(H$	$\lor$	$S)$	H	$\neg$	S
T	T	T	T	T	T	F	T
T	F	T	T	F	T	T	F
F	T	F	T	T	F	F	T
F	F	F	F	F	F	T	F

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

An invalid argument

$$
\begin{array}{l}
H \vee S \\
H \\
\therefore \neg S
\end{array}
$$

H	S	$(H$	$\vee$	$S)$	H	$\neg$	S
T	**T**	T	**T**	T	**T**	**F**	T
T	**F**	T	**T**	F	**T**	**T**	F
F	**T**	F	**T**	T	**F**	**F**	T
F	**F**	F	**F**	F	**F**	**T**	F

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

An invalid argument

$$
\begin{array}{l}
H \vee S \\
H \\
\therefore \neg S
\end{array}
$$

H	S	$(H$	$\vee$	$S)$	H	$\neg$	S	
T	**T**	T	**T**	T	**T**	**F**	T	$\leftarrow$
T	**F**	T	**T**	F	**T**	**T**	F	
F	**T**	F	**T**	T	**F**	**F**	T	
F	**F**	F	**F**	F	**F**	**T**	F	

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

An invalid argument

$$
\begin{array}{l}
H \vee S \\
H \\
\therefore \neg S
\end{array}
$$

H	S	$(H$	$\vee$	$S)$	H	$\neg$	S	
T	T	T	T	T	T	F	T	✗
T	F	T	T	F	T	T	F	
F	T	F	T	T	F	F	T	
F	F	F	F	F	F	T	F	

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

An invalid argument

$$
\begin{array}{l}
H \vee S \\
H \\
\therefore \neg S
\end{array}
$$

H	S	$(H$	$\vee$	$S)$	H	$\neg$	S	
T	**T**	T	**T**	T	**T**	**F**	T	✗
T	**F**	T	**T**	F	**T**	**T**	F	←
F	**T**	F	**T**	T	**F**	**F**	T	
F	**F**	F	**F**	F	**F**	**T**	F	

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

An invalid argument

$$
\begin{array}{l}
H \lor S \\
H \\
\therefore \neg S
\end{array}
$$

H	S	$(H$	$\lor$	$S)$	H	$\neg$	S	
T	**T**	T	**T**	T	**T**	**F**	T	✗
T	**F**	T	**T**	F	**T**	**T**	F	✓
F	**T**	F	**T**	T	**F**	**F**	T	←
F	**F**	F	**F**	F	**F**	**T**	F	

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

An invalid argument

$$
\begin{array}{l}
H \lor S \\
H \\
\therefore \neg S
\end{array}
$$

H	S	$(H$	$\lor$	$S)$	H	$\neg$	S	
T	**T**	T	**T**	T	**T**	**F**	T	✗
T	**F**	T	**T**	F	**T**	**T**	F	✓
F	**T**	F	**T**	T	**F**	**F**	T	✓
F	**F**	F	**F**	F	**F**	**T**	F	←

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

An invalid argument

$$
\begin{array}{l}
H \lor S \\
H \\
\therefore \neg S
\end{array}
$$

H	S	$(H$	$\lor$	$S)$	H	$\neg$	S	
T	**T**	T	**T**	T	**T**	**F**	T	✗
T	**F**	T	**T**	F	**T**	**T**	F	✓
F	**T**	F	**T**	T	**F**	**F**	T	✓
F	**F**	F	**F**	F	**F**	**T**	F	✓

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

Lecture 4

Monday, January 20, 2020

More Connectives

More connectives

Exclusive or

Example

Mandy enjoys hiking or skiing but not both.

.

Exclusive or

Example

Mandy enjoys hiking or skiing but not both.

[either H or S]

.

Exclusive or

Example

Mandy enjoys hiking or skiing but not both.

[either H or S]

.

Exclusive or

Example

Mandy enjoys hiking or skiing but not both.

[either H or S]
[it is not the case that [both H and S]].

Exclusive or

Example

Mandy enjoys hiking or skiing but not both.

[either H or S]
[it is not the case that [both H and S]].

Exclusive or

Example

Mandy enjoys hiking or skiing but not both.

Both [either H or S] and
[it is not the case that [both H and S]].

Exclusive or

Paraphrase sentences containing "either A or B but not both" using "**both [either A or B] and**
[it is not the case that [both A and B]]"

Example

Mandy enjoys hiking or skiing but not both.

Both [either H or S] and
[it is not the case that [both H and S]].

Exclusive or

Paraphrase sentences containing "either A or B but not both" using "**both [either A or B] and**
[it is not the case that [both A and B]]"

Example

Mandy enjoys hiking or skiing but not both.

Both [either H or S] and
[it is not the case that [both H and S]].

$((H \vee S) \wedge \neg(H \wedge S))$

Neither ... nor ...

Example

Mandy enjoys neither hiking nor skiing.

Neither ... nor ...

Paraphrase sentences containing "neither A nor B" using
"**both [it is not the case that A] and**
[it is not the case that B]"

Example

Mandy enjoys neither hiking nor skiing.

Both [it is not the case that H] and
[it is not the case that S].

Neither ... nor ...

Paraphrase sentences containing "neither A nor B" using
"**both [it is not the case that A] and**
[it is not the case that B]"

Example

Mandy enjoys neither hiking nor skiing.

Both [it is not the case that H] and
[it is not the case that S].

$(\neg H \wedge \neg S)$

A Logic Puzzle

- Every card has a letter on one side and a number on the other side
- You're a card inspector tasked with making sure that cards satisfy this quality standard:

  If a card has an even number on one side, then it has a vowel on the other.

A Logic Puzzle

Which card(s) do you have to turn over to make sure that:

If a card has an even number on one side, then it has a vowel on the other.

E	K	3	4
(1)	(2)	(3)	(4)

Another logic puzzle

- At an all-ages event where everyone has a drink
- You know how old some of the people are, and you can tell what some of them are drinking
- You're tasked with making sure that the following rule is followed:

  If a person is drinking alcohol, then they are at least 18 years old.

Another logic puzzle

Which of these people do you have to check (age or drink) to ensure that:

> *If a person is drinking alcohol, then they must be at least 18 years old*

22 years	16 years	drinks pop	drinks beer
(1)	(2)	(3)	(4)

Truth conditions of conditionals

$$\text{If } \underbrace{\text{X is drinking alcohol}}_{A}\text{, then } \underbrace{\text{X is over 18}}_{B}$$

- "If A, then B" can only be **false** if:
  - A is **true**: we check age if X is drinking beer (A true), not if drinking pop; **and**
  - B is **false**: we check drink if X underage (B false), not if over 18 (B true)
- "If A, then B" is true if:
  - A is **false** (we don't check people drinking pop); **or**
  - B is **true** (we don't card if X is over 18);
  - (or both)

The material conditional $\rightarrow$

$\mathscr{A}$	$\mathscr{B}$	$(\mathscr{A} \rightarrow \mathscr{B})$
T	T	T
T	F	F
F	T	T
F	F	T

If and only if

- Sue drinks beer (A) if she is over 18 (B).

$$B \to A$$

- Sue drinks beer (A) only if she is over 18 (B)

$$A \to B$$

Unless

Which of these people do you have to check (age or drink) to ensure that:

People are drinking pop unless they are over 18.

22 years	16 years	drinks pop	drinks beer
(1)	(2)	(3)	(4)

Unless

$$\underbrace{\text{X is drinking pop}}_{A}, \text{ unless } \underbrace{\text{X is over 18}}_{B}$$

- "A unless B" can only be **false** if:
  - A is **false**
    (we check age if person is drinking beer), **and**
  - B is **false**
    (we check drink if person not at least 18)
- "A unless B" is true (test OK) if:
  - A is false, and B is true;
  - A is true, and B is false;
  - A is true and B is true
- "A unless B" is equivalent to
  - "A if not B" ($\neg B \rightarrow A$)
  - "either A or B" ($A \vee B$)

Unless

Treat "**unless**" the same way you would treat "or"

Example

Mandy enjoys hiking unless Sanjiv is from Calgary.

$(H \vee C)$

Large truth tables

Example

Valid or invalid?

Sara is from Calgary or Edmonton.

Mariusz is from Calgary unless he enjoys hiking.

If Mariusz is from Calgary, Sara isn't.

Neither Sara nor Mariusz enjoy hiking.

$\therefore$ Sara is from Edmonton.

Example

Valid or invalid?

Sara is from Calgary or Edmonton.

Either [Sara is from Calgary] or [Sara is from Edmonton].

Mariusz is from Calgary unless he enjoys hiking.

If Mariusz is from Calgary, Sara isn't.

Neither Sara nor Mariusz enjoy hiking.

$\therefore$ Sara is from Edmonton.

Example

Valid or invalid?

Sara is from Calgary or Edmonton.

Either [Sara is from Calgary] or [Sara is from Edmonton].

Mariusz is from Calgary unless he enjoys hiking.

Either [Mariusz is from Calgary] or [Mariusz enjoys hiking].

If Mariusz is from Calgary, Sara isn't.

Neither Sara nor Mariusz enjoy hiking.

$\therefore$ Sara is from Edmonton.

Example

Valid or invalid?

Sara is from Calgary or Edmonton.

Either [Sara is from Calgary] or [Sara is from Edmonton].

Mariusz is from Calgary unless he enjoys hiking.

Either [Mariusz is from Calgary] or [Mariusz enjoys hiking].

If Mariusz is from Calgary, Sara isn't.

If [Mariusz is from Calgary] then [it is not the case that [Sara is from Calgary]].

Neither Sara nor Mariusz enjoy hiking.

$\therefore$ Sara is from Edmonton.

Example

Valid or invalid?

Sara is from Calgary or Edmonton.

Either [Sara is from Calgary] or [Sara is from Edmonton].

Mariusz is from Calgary unless he enjoys hiking.

Either [Mariusz is from Calgary] or [Mariusz enjoys hiking].

If Mariusz is from Calgary, Sara isn't.

If [Mariusz is from Calgary] then [it is not the case that [Sara is from Calgary]].

Neither Sara nor Mariusz enjoy hiking.

Both [it is not the case that [Sara enjoys hiking]] and [it is not the case that [Mariusz enjoys hiking]].

$\therefore$ Sara is from Edmonton.

Example

Valid or invalid?

Sara is from Calgary or Edmonton.

$(C \vee E)$

Mariusz is from Calgary unless he enjoys hiking.

$(A \vee M)$

If Mariusz is from Calgary, Sara isn't.

$(A \rightarrow \neg C)$

Neither Sara nor Mariusz enjoy hiking.

$(\neg S \wedge \neg M)$

$\therefore$ Sara is from Edmonton.

$\therefore\ E$.

Large truth tables

- For arguments with n sentence letters, there are 2^n possible valuations
  - A single letter A can be **T** or **F**: $2^1 = 2$ valuations.
  - For two letters A, B: B can be **T** or **F** for every possible valuation (2) of A: $2 \times 2 = 2^2 = 4$ valuations
  - For three letters A, B, C: C can be **T** or **F** for every possible valuation (4) of A and B: $2 \times 4 = 2^3 = 8$ valuations
  - Etc.
- In the ith reference column, alternate **T** and **F** every 2^{n-i} lines

Example (simplified)

Sara is from Calgary or Edmonton.
Mariusz is from Calgary unless he enjoys hiking.
If Mariusz is from Calgary, Sara isn't.
Mariusz doesn't enjoy hiking.
$\therefore$ Sara is from Edmonton.

$$
\begin{aligned}
& C \vee E \\
& A \vee M \\
& A \rightarrow \neg C \\
& \neg M \\
\therefore & E
\end{aligned}
$$

A complex truth table

3 sentence letters A, C, E: $2^3 = 8$ lines

	A	C	E	...
1				...
2				...
3				...
4				...
5				...
6				...
7				...
8				...

A complex truth table

3 sentence letters A, C, E: $2^3 = 8$ lines

	A	C	E	...
1				...
2				...
3				...
4				...
5				...
6				...
7				...
8				...

↑
alternate every ...
4
rows

A complex truth table

3 sentence letters A, C, E: $2^3 = 8$ lines

	A	C	E	...
1	**T**			...
2	**T**			...
3	**T**			...
4	**T**			...
5	**F**			...
6	**F**			...
7	**F**			...
8	**F**			...

↑

alternate every ... 2 rows

A complex truth table

3 sentence letters A, C, E: $2^3 = 8$ lines

	A	C	E	...
1	**T**	**T**		...
2	**T**	**T**		...
3	**T**	**F**		...
4	**T**	**F**		...
5	**F**	**T**		...
6	**F**	**T**		...
7	**F**	**F**		...
8	**F**	**F**		...

↑

alternate every ... 1 rows

A complex truth table

3 sentence letters A, C, E: $2^3 = 8$ lines

	A	C	E	...
1	**T**	**T**	**T**	...
2	**T**	**T**	**F**	...
3	**T**	**F**	**T**	...
4	**T**	**F**	**F**	...
5	**F**	**T**	**T**	...
6	**F**	**T**	**F**	...
7	**F**	**F**	**T**	...
8	**F**	**F**	**F**	...

alternate every ...
1
rows

A	C	E	M	$C \vee E$	$A \vee M$	$A \rightarrow \neg C$	$\neg M$	E
T	T	T	T					
T	T	T	F					
T	T	F	T					
T	T	F	F					
T	F	T	T					
T	F	T	F					
T	F	F	T					
T	F	F	F					
F	T	T	T					
F	T	T	F					
F	T	F	T					
F	T	F	F					
F	F	T	T					
F	F	T	F					
F	F	F	T					
F	F	F	F					

A	C	E	M	$C \vee E$		$A \vee M$		$A \rightarrow \neg C$		$\neg M$	E
T	T	T	T	T	T	T	T	T	T	T	T
T	T	T	F	T	T	T	F	T	T	F	T
T	T	F	T	T	F	T	T	T	T	T	F
T	T	F	F	T	F	T	F	T	T	F	F
T	F	T	T	F	T	T	T	T	F	T	T
T	F	T	F	F	T	T	F	T	F	F	T
T	F	F	T	F	F	T	T	T	F	T	F
T	F	F	F	F	F	T	F	T	F	F	F
F	T	T	T	T	T	F	T	F	T	T	T
F	T	T	F	T	T	F	F	F	T	F	T
F	T	F	T	T	F	F	T	F	T	T	F
F	T	F	F	T	F	F	F	F	T	F	F
F	F	T	T	F	T	F	T	F	F	T	T
F	F	T	F	F	T	F	F	F	F	F	T
F	F	F	T	F	F	F	T	F	F	T	F
F	F	F	F	F	F	F	F	F	F	F	F

A	C	E	M	$C \vee E$	$A \vee M$	$A \rightarrow \neg C$	$\neg M$	E
T	T	T	T	T T T	T T T	T F F T	F T	T
T	T	T	F	T T T	T T F	T F F T	T F	T
T	T	F	T	T T F	T T T	T F F T	F T	F
T	T	F	F	T T F	T T F	T F F T	T F	F
T	F	T	T	F T T	T T T	T T T F	F T	T
T	F	T	F	F T T	T T F	T T T F	T F	T
T	F	F	T	F F F	T T T	T T T F	F T	F
T	F	F	F	F F F	T T F	T T T F	T F	F
F	T	T	T	T T T	F T T	F T F T	F T	T
F	T	T	F	T T T	F F F	F T F T	T F	T
F	T	F	T	T T F	F T T	F T F T	F T	F
F	T	F	F	T T F	F F F	F T F T	T F	F
F	F	T	T	F T T	F T T	F T T F	F T	T
F	F	T	F	F T T	F F F	F T T F	T F	T
F	F	F	T	F F F	F T T	F T T F	F T	F
F	F	F	F	F F F	F F F	F T T F	T F	F

A	C	E	M	$C \vee E$	$A \vee M$	$A \rightarrow \neg C$	$\neg M$	E
T	T	T	T	T T T	T T T	T F F T	F T	T
T	T	T	F	T T T	T T F	T F F T	T F	T
T	T	F	T	T T F	T T T	T F F T	F T	F
T	T	F	F	T T F	T T F	T F F T	T F	F
T	F	T	T	F T T	T T T	T T T F	F T	T
T	F	T	F	F T T	T T F	T T T F	T F	T
T	F	F	T	F F F	T T T	T T T F	F T	F
T	F	F	F	F F F	T T F	T T T F	T F	F
F	T	T	T	T T T	F T T	F T F T	F T	T
F	T	T	F	T T T	F F F	F T F T	T F	T
F	T	F	T	T T F	F T T	F T F T	F T	F
F	T	F	F	T T F	F F F	F T F T	T F	F
F	F	T	T	F T T	F T T	F T T F	F T	T
F	F	T	F	F T T	F F F	F T T F	T F	T
F	F	F	T	F F F	F T T	F T T F	F T	F
F	F	F	F	F F F	F F F	F T T F	T F	F

A	C	E	M	$C \vee E$	$A \vee M$	$A \to \neg C$	$\neg M$	E
T	T	T	T	T T T	T T T	T F F T	F T	T
T	T	T	F	T T T	T T F	T F F T	T F	T
T	T	F	T	T T F	T T T	T F F T	F T	F
T	T	F	F	T T F	T T F	T F F T	T F	F
T	F	T	T	F T T	T T T	T T T F	F T	T
T	F	T	F	F T T	T T F	T T T F	T F	T
T	F	F	T	F F F	T T T	T T T F	F T	F
T	F	F	F	F F F	T T F	T T T F	T F	F
F	T	T	T	T T T	F T T	F T F T	F T	T
F	T	T	F	T T T	F F F	F T F T	T F	T
F	T	F	T	T T F	F T T	F T F T	F T	F
F	T	F	F	T T F	F F F	F T F T	T F	F
F	F	T	T	F T T	F T T	F T T F	F T	T
F	F	T	F	F T T	F F F	F T T F	T F	T
F	F	F	T	F F F	F T T	F T T F	F T	F
F	F	F	F	F F F	F F F	F T T F	T F	F

Every valuation makes at least one premise false, or makes the conclusion true: the argument is valid.

Lecture 5

Wednesday, January 22, 2020

Logical notions in TFL

Logical notions of TFL
Chapter 11

Validity of arguments

Definition

An argument is **valid in TFL** iff every valuation either makes one or more of the premises false or it makes the conclusion true.

An argument is **invalid in TFL** iff at least one valuation makes all the premises true and it makes the conclusion false.

Entailment

Definition

Sentences $\mathscr{A}_1, \ldots, \mathscr{A}_n$ **entail** a sentence $\mathscr{B}$ iff every valuation either makes at least one of $\mathscr{A}_1, \ldots, \mathscr{A}_n$ false or makes $\mathscr{B}$ true.

In that case we write $\mathscr{A}_1, \ldots, \mathscr{A}_n \vDash \mathscr{B}$.

We have:

$$\mathscr{A}_1, \ldots, \mathscr{A}_n \vDash \mathscr{B} \text{ iff the argument } \mathscr{A}_1, \ldots, \mathscr{A}_n \therefore \mathscr{B} \text{ is valid.}$$

Entailment

Does $\neg(\neg A \vee \neg B), A \to \neg C \vDash A \to (B \to C)$?

Entailment

A	B	C	$\neg$	$(\neg$	A	$\vee$	$\neg$	$B)$	A	$\rightarrow$	$\neg$	C	A	$\rightarrow$	$(B$	$\rightarrow$	$C)$
T	T	T	T	F	T	F	F	T	T	F	F	T	T	T	T	T	T
T	T	F	T	F	T	F	F	T	T	T	T	F	T	F	T	F	F
T	F	T	F	F	T	T	T	F	T	F	F	T	T	T	F	T	T
T	F	F	F	F	T	T	T	F	T	T	T	F	T	T	F	T	F
F	T	T	F	T	F	T	F	T	F	T	F	T	F	T	T	T	T
F	T	F	F	T	F	T	F	T	F	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	T	F	F	T	F	T	F	T	F	T	T
F	F	F	F	T	F	T	T	F	F	T	T	F	F	T	F	T	F

Entailment

A	B	C	$\neg$	$(\neg A$	$\vee$	$\neg B)$		A	$\rightarrow$	$\neg C$		A	$\rightarrow$	$(B$	$\rightarrow$	$C)$	
T	T	T	T	F	T	F	F	T	T	F	T	T	T	T	T	T	
T	T	F	T	F	T	F	F	T	T	T	F	T	F	T	F	F	←
T	F	T	F	F	T	T	T	F	T	F	T	T	T	F	T	T	
T	F	F	F	F	T	T	T	F	T	T	F	T	T	F	T	F	
F	T	T	F	T	F	T	F	T	F	T	T	F	T	T	T	T	
F	T	F	F	T	F	T	F	T	F	T	F	F	T	T	F	F	
F	F	T	F	T	F	T	T	F	F	T	T	F	T	F	T	T	
F	F	F	F	T	F	T	T	F	F	T	F	F	T	F	T	F	

Equivalent sentences

Definition

Two sentences $\mathscr{A}$ and $\mathscr{B}$ are **equivalent in TFL** iff every valuation either makes both $\mathscr{A}$ and $\mathscr{B}$ true or it makes both $\mathscr{A}$ and $\mathscr{B}$ false.

In other words: $\mathscr{A}$ and $\mathscr{B}$ agree in truth value, for every valuation.

Equivalent sentences

A	B	$\neg$	A	$\vee$	B	A	$\rightarrow$	B
T	T		T		T	T		T
T	F		T		F	T		F
F	T		F		T	F		T
F	F		F		F	F		F

Equivalent sentences

A	B	$\neg$	A	$\vee$	B	A	$\rightarrow$	B
T	T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	F	F
F	T	T	F	T	T	F	T	T
F	F	T	F	T	F	F	T	F

Equivalent sentences

A	B	$\neg$	A	$\vee$	B	A	$\rightarrow$	B
T	T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	F	F
F	T	T	F	T	T	F	T	T
F	F	T	F	T	F	F	T	F

Equivalence and entailment

Fact

If $\mathscr{A}$ and $\mathscr{B}$ are equivalent, then $\mathscr{A} \vDash \mathscr{B}$ (and $\mathscr{B} \vDash \mathscr{A}$).

Equivalence and entailment

Fact

If $\mathscr{A}$ and $\mathscr{B}$ are equivalent, then $\mathscr{A} \vDash \mathscr{B}$ (and $\mathscr{B} \vDash \mathscr{A}$).

Proof

- Look at any valuation: it makes $\mathscr{A}$ true or false.
- If $\mathscr{A}$ is false, the valuation is not a counterexample.
- If $\mathscr{A}$ is true, $\mathscr{B}$ is also true (since $\mathscr{A}$ and $\mathscr{B}$ agree in truth value on every valuation).
- So if $\mathscr{A}$ is true, the valuation is not a counterexample.
- So, no valuation can be a counterexample to $\mathscr{A} \vDash \mathscr{B}$.

Tautologies

Definition

A sentence $\mathscr{A}$ is a **tautology** iff it is true on every valuation.

A	A	$\rightarrow$	A
T	**T**	**T**	**T**
F	**F**	**T**	**F**

Joint satisfiability

Definition

Sentences $\mathscr{A}_1, \ldots, \mathscr{A}_n$ are **jointly satisfiable** in TFL if there is at least one valuation that makes all of them true.

If they are not satisfiable, we say they are **jointly unsatisfiable**.

$A \vee B$, $\neg A$, B are jointly satisfiable.

$A \vee B$, $\neg A$, $\neg B$ are unsatisfiable.

Application

A, B, C, D: Amir, Betty, Ching, Dana are in the boat.

Amir won't go without Ching.

Ching only goes if at least one of Betty and Dana goes too.

Amir and Dana can't be in the boat together.

Application

A, B, C, D: Amir, Betty, Ching, Dana are in the boat.

Amir won't go without Ching.

$$A \to C$$

Ching only goes if at least one of Betty and Dana goes too.

Amir and Dana can't be in the boat together.

Application

A, B, C, D: Amir, Betty, Ching, Dana are in the boat.

Amir won't go without Ching.

$$A \to C$$

Ching only goes if at least one of Betty and Dana goes too.

$$C \to (B \vee D)$$

Amir and Dana can't be in the boat together.

Application

A, B, C, D: Amir, Betty, Ching, Dana are in the boat.

Amir won't go without Ching.

$$A \to C$$

Ching only goes if at least one of Betty and Dana goes too.

$$C \to (B \vee D)$$

Amir and Dana can't be in the boat together.

$$\neg(A \wedge D)$$

$$A \to \neg D$$

$$\neg A \vee \neg D$$

Another application

A, B, C, D: package A, B, C, D is installed.

Package A depends on package C.

Package C requires either package B or D.

Package A is incompatible with package D.

Another application

A, B, C, D: package A, B, C, D is installed.

Package A depends on package C.

$$A \to C$$

Package C requires either package B or D.

$$C \to (B \vee D)$$

Package A is incompatible with package D.

$$\neg(A \wedge D)$$

$$A \to \neg D$$

$$\neg A \vee \neg D$$

Dependency resolution by SAT checking

Can you send **Amir** in the boat?

Can package **A** be installed ?

Same as: Are these sentences jointly satisfiable?

$$\mathbf{A}$$

$$A \to C$$

$$C \to (B \vee D)$$

$$\neg(A \wedge D)$$

(Exercise: construct a complete truth table. Which valuations, if any, satisfy all four sentences?)

Dependency resolution by SAT checking

Can you send **Amir without Betty** in the boat?

Can package **A** be installed **without installing B**?

Same as: Are these sentences jointly satisfiable?

$$A \wedge \neg B$$

$$A \rightarrow C$$

$$C \rightarrow (B \vee D)$$

$$\neg(A \wedge D)$$

(Exercise: construct a complete truth table. Which valuations, if any, satisfy all four sentences?)

Complexity of logical testing

- Testing for validity, satisfiability, tautology, etc., requires making a complete truth table
  - Testing for validity requires checking **every** valuation.
  - Testing for satisfiability requires finding **at least one** valuation.
- If there are n sentence letters, there are 2^n valuations to check.
- Computer scientists have yet to find a method that can (always) do this faster than truth tables (“P vs NP problem”).

Lecture 6

Friday, January 24, 2020

Introduction to proofs. Rules for $\wedge$

Proofs in TFL
Chapters 14–16

Showing that an argument is valid

- Construct a truth table; verify there is no valuation where premises are true and conclusion is false.
- Truth tables can get very large very quickly.
- E.g., the example argument

$$C \vee E, A \vee M, A \rightarrow \neg C, \neg S \wedge \neg M \therefore E$$

  requires 32 lines and 608 individual truth values.
- Is there a better way?

Proofs

- Idea: work our way from premises to conclusion using steps we know are entailed by the premises.
- For instance:
  - From “Neither Sara nor Mariusz enjoys hiking” we can conclude “Mariusz doesn’t enjoy hiking.”
  - From “Either Marius is from Calgary or he enjoys hiking” and “Mariusz doesn’t enjoy hiking” we can conclude “Mariusz is from Calgary” (Disjunctive syllogism DS).
  - etc.
- If we manage to work from the premises to the conclusion in this way, we know that the argument must be valid.

An informal proof

Our argument

1. Sara is from Calgary or Edmonton.
2. Mariusz is from Calgary unless he enjoys hiking.
3. If Mariusz is from Calgary, Sara isn't.
4. Neither Sara nor Mariusz enjoy hiking.

$\therefore$ Sara is from Edmonton.

5. Mariusz doesn't enjoy hiking (from 4).

An informal proof

Our argument

1. Sara is from Calgary or Edmonton.
2. Mariusz is from Calgary unless he enjoys hiking.
3. If Mariusz is from Calgary, Sara isn't.
4. Neither Sara nor Mariusz enjoy hiking.

$\therefore$ Sara is from Edmonton.

5. Mariusz doesn't enjoy hiking (from 4).
6. Mariusz is from Calgary (from 2 and 5).

An informal proof

Our argument

1. Sara is from Calgary or Edmonton.
2. Mariusz is from Calgary unless he enjoys hiking.
3. If Mariusz is from Calgary, Sara isn't.
4. Neither Sara nor Mariusz enjoy hiking.

$\therefore$ Sara is from Edmonton.

5. Mariusz doesn't enjoy hiking (from 4).
6. Mariusz is from Calgary (from 2 and 5).
7. Sara isn't from Calgary (from 3 and 6).

An informal proof

Our argument

1. Sara is from Calgary or Edmonton.
2. Mariusz is from Calgary unless he enjoys hiking.
3. If Mariusz is from Calgary, Sara isn't.
4. Neither Sara nor Mariusz enjoy hiking.

$\therefore$ Sara is from Edmonton.

5. Mariusz doesn't enjoy hiking (from 4).
6. Mariusz is from Calgary (from 2 and 5).
7. Sara isn't from Calgary (from 3 and 6).
8. Sara is from Edmonton (from 1 and 7).

A more formal proof

Our argument

1. $C \vee E$
2. $A \vee M$
3. $A \rightarrow \neg C$
4. $\neg S \wedge \neg M$

$\therefore E$

5. $\neg M$ (from 4, since $\mathcal{P} \wedge \mathcal{Q} \vDash \mathcal{Q}$)

A more formal proof

Our argument

1. $C \vee E$
2. $A \vee M$
3. $A \rightarrow \neg C$
4. $\neg S \wedge \neg M$

$\therefore E$

5. $\neg M$ (from 4, since $\mathcal{P} \wedge \mathcal{Q} \vDash \mathcal{Q}$)
6. A (from 2 and 5, since $\mathcal{P} \vee \mathcal{Q}, \neg \mathcal{Q} \vDash \mathcal{P}$)

A more formal proof

Our argument

1. $C \vee E$
2. $A \vee M$
3. $A \to \neg C$
4. $\neg S \wedge \neg M$

$\therefore E$

5. $\neg M$ (from 4, since $\mathcal{P} \wedge \mathcal{Q} \vDash \mathcal{Q}$)
6. A (from 2 and 5, since $\mathcal{P} \vee \mathcal{Q}, \neg\mathcal{Q} \vDash \mathcal{P}$)
7. $\neg C$ (from 3 and 6, since $\mathcal{P} \to \mathcal{Q}, \mathcal{P} \vDash \mathcal{Q}$)

A more formal proof

Our argument

1. $C \vee E$
2. $A \vee M$
3. $A \rightarrow \neg C$
4. $\neg S \wedge \neg M$

$\therefore E$

5. $\neg M$ (from 4, since $\mathcal{P} \wedge \mathcal{Q} \vDash \mathcal{Q}$)
6. A (from 2 and 5, since $\mathcal{P} \vee \mathcal{Q}, \neg\mathcal{Q} \vDash \mathcal{P}$)
7. $\neg C$ (from 3 and 6, since $\mathcal{P} \rightarrow \mathcal{Q}, \mathcal{P} \vDash \mathcal{Q}$)
8. E (from 1 and 7, since $\mathcal{P} \vee \mathcal{Q}, \neg\mathcal{P} \vDash \mathcal{Q}$)

Formal proofs

- Numbered lines containing sentences of TFL
- A line may be a **premise**
- If it's not a premise, it must be **justified**
- Justification involves
  - a **rule**
  - prior lines (referred to by line number)
- But: what's a rule?

Rules of natural deduction

- Rules should be ...
  - **Simple**: cite just a few lines as justification
  - **Obvious**: justified line should clearly be entailed by justifications
  - **Schematic**: can be described just by **forms** of sentences involved
  - **Few in number**: want to make do with just a handful
- We'll have two rules per connective: an **introduction** and an **elimination** rule
- They'll either justify (say) $\mathcal{P} \wedge \mathcal{Q}$ ($\wedge$I), or use it to justify something else ($\wedge$E)

Eliminating $\wedge$

- What can we **justify using** $\mathcal{P} \wedge \mathcal{Q}$?
- We have:

$$\mathcal{P} \wedge \mathcal{Q} \vDash \mathcal{P}$$

$$\mathcal{P} \wedge \mathcal{Q} \vDash \mathcal{Q}$$

- Already used above to get $\neg M$ from $\neg S \wedge \neg M$, i.e., from "Neither Sara nor Mariusz enjoys hiking" we can conclude "Mariusz doesn't enjoy hiking"
- (Role of $\mathcal{P}$ played by $\neg S$ and $\mathcal{Q}$ played by $\neg M$)

Introducing $\wedge$

- What do we **need to justify** $\mathscr{P} \wedge \mathbb{Q}$?
- We need both $\mathscr{P}$ and $\mathbb{Q}$:

$$\mathscr{P}, \mathbb{Q} \vDash \mathscr{P} \wedge \mathbb{Q}$$

- For instance, if we have "Sara doesn't enjoy hiking" and also "Mariusz doesn't enjoy hiking" we can conclude "Neither Sara nor Mariusz enjoys hiking"
- (Role of $\mathscr{P}$ played by $\neg S$ and $\mathbb{Q}$ played by $\neg M$: $\neg S, \neg M \vDash \neg S \wedge \neg M$)

Rules for $\wedge$

$$\begin{array}{r|ll} m & \mathcal{P} & \\ n & \mathcal{Q} & \\ & \mathcal{P} \wedge \mathcal{Q} & \wedge\mathrm{I}\ m,\ n \end{array}$$

$$\begin{array}{r|ll} m & \mathcal{P} \wedge \mathcal{Q} & \\ & \mathcal{P} & \wedge\mathrm{E}\ m \end{array}$$

$$\begin{array}{r|ll} m & \mathcal{P} \wedge \mathcal{Q} & \\ & \mathcal{Q} & \wedge\mathrm{E}\ m \end{array}$$

We'll illustrate using the exercises in here.

1	$A \wedge B$	
2	A	$\wedge$E 1
3	B	$\wedge$E 1
4	$B \wedge A$	$\wedge$I 2, 3

1	$A \wedge (B \wedge C)$	
2	A	$\wedge$E 1
3	$B \wedge C$	$\wedge$E 1
4	C	$\wedge$E 3
5	$A \wedge C$	$\wedge$I 2, 4

Lecture 7

Monday, January 27, 2020

Proof with $\rightarrow$. Subproofs

Eliminating $\rightarrow$

- What can we **justify using** $\mathcal{P} \rightarrow \mathcal{Q}$?
- We used the conditional "If Mariusz is from Calgary, Sara isn't" to justify "Sara isn't from Calgary"
- What is the general rule? What can we justify using $\mathcal{P} \rightarrow \mathcal{Q}$? What do we need in addition to $\mathcal{P} \rightarrow \mathcal{Q}$?

Eliminating $\rightarrow$

- What can we **justify using** $\mathcal{P} \rightarrow \mathcal{Q}$?
- We used the conditional "If Mariusz is from Calgary, Sara isn't" to justify "Sara isn't from Calgary"
- What is the general rule? What can we justify using $\mathcal{P} \rightarrow \mathcal{Q}$? What do we need in addition to $\mathcal{P} \rightarrow \mathcal{Q}$?
- The principle is **modus ponens**:

$$\mathcal{P} \rightarrow \mathcal{Q}, \mathcal{P} \vDash \mathcal{Q}$$

- In inference from $A \rightarrow \neg C$ and A to $\neg C$, role of $\mathcal{P}$ is played by A and role of $\mathcal{Q}$ by $\neg C$.

Elimination rule for $\to$

m	$\mathscr{P} \to \mathscr{Q}$	
n	$\mathscr{P}$	
	$\mathscr{Q}$	$\to$E m, n

We'll illustrate using the exercise here: show that $A \wedge B, A \to C, B \to D \models C \wedge D$.

1	$A \wedge B$	
2	$A \to C$	
3	$B \to D$	
4	A	$\wedge$E 1
5	C	$\to$E 2, 4
6	B	$\wedge$E 1
7	D	$\to$E 3, 6
8	$C \wedge D$	$\wedge$I 5, 7

Introducing $\rightarrow$

- How do we justify a conditional? What should we require for a proof of $\mathcal{P} \rightarrow \mathcal{Q}$ (say, from some premise $\mathcal{R}$)?
- We need a proof that shows that $\mathcal{R} \vDash \mathcal{P} \rightarrow \mathcal{Q}$.
- Idea: show instead that $\mathcal{R}, \mathcal{P} \vDash \mathcal{Q}$.
- The conditional $\rightarrow$ no longer appears, so this seems easier.
- It's a good move, because if $\mathcal{R}, \mathcal{P} \vDash \mathcal{Q}$ then $\mathcal{R} \vDash \mathcal{P} \rightarrow \mathcal{Q}$.

Justifying $\to$I

Fact

If $\mathcal{R}, \mathcal{P} \vDash \mathcal{Q}$ then $\mathcal{R} \vDash \mathcal{P} \to \mathcal{Q}$.

- If $\mathcal{R}, \mathcal{P} \vDash \mathcal{Q}$ then every valuation makes one of $\mathcal{R}, \mathcal{P}$ false or it makes $\mathcal{Q}$ true.
- Let's show that no valuation is a counterexample to $\mathcal{R} \vDash \mathcal{P} \to \mathcal{Q}$:
  1. If it makes $\mathcal{R}$ false, it's not a counterexample to $\mathcal{R} \vDash \mathcal{P} \to \mathcal{Q}$.
  2. If it makes $\mathcal{P}$ false, it makes $\mathcal{P} \to \mathcal{Q}$ true, so it's not a counterexample.
  3. If it makes $\mathcal{Q}$ true, it also makes $\mathcal{P} \to \mathcal{Q}$ true, so it's not a counterexample.
- So, there are no counterexamples to $\mathcal{R} \vDash \mathcal{P} \to \mathcal{Q}$.

Subproofs

- So we want to justify $\mathcal{P} \to \mathcal{Q}$ by giving a proof of $\mathcal{Q}$ from $\mathcal{P}$ as a premise.
- How to do this in a proof? We can't just add something as a premise and then remove it later!
- Solution: add $\mathcal{P}$ as a premise in the middle, and keep track of what depends on that premise (say, by a indenting and vertical line)
- Once we're done (have proved $\mathcal{Q}$), close this "subproof".
- Justification of $\mathcal{P} \to \mathcal{Q}$ is the **entire** subproof.
- **Important**: nothing **inside** a subproof is available outside as a justification (it depends on the assumption)

Introduction rule for $\to$

$$
\begin{array}{l|l}
m & \begin{array}{|l} \mathscr{P} \\ \hline \vdots \\ \\ \mathbb{Q} \end{array} \\
n & \\
 & \mathscr{P} \to \mathbb{Q} \qquad \to\mathrm{I}\ m\text{–}n
\end{array}
$$

We'll illustrate using the exercises here

- Show: $A \to B, B \to C \models A \to C$.
- Show: $A \to (B \to C) \models (A \wedge B) \to (A \wedge C)$

1	$A \to B$	
2	$B \to C$	
3	$\quad A$	
4	$\quad B$	$\to$E 1, 3
5	$\quad C$	$\to$E 2, 4
6	$A \to C$	$\to$I 3–5

1	$A \to (B \to C)$	
2	$\quad A \wedge B$	
3	$\quad A$	$\wedge$E 2
4	$\quad B \to C$	$\to$E 1, 3
5	$\quad B$	$\wedge$E 2
6	$\quad C$	$\to$E 4, 5
7	$\quad A \wedge C$	$\wedge$I 3, 6
8	$(A \wedge B) \to (A \wedge C)$	$\to$I 2–7

Reiteration

$\mathscr{P} \models \mathscr{P}$ so this is a good rule:

m	$\mathscr{P}$	
k	$\mathscr{P}$	R m

Uses of reiteration:

- Proof of $A \models A$.
- Proof that $A \to (B \to A)$ is a tautology.

1	$\quad A$	
2	$\quad A$	R 1
3	$A \to A$	$\to$I 1-2

1	A	
2	B	
3	A	R 1
4	$B \to A$	$\to$I 2–3
5	$A \to (B \to A)$	$\to$I 1–4

Lecture 8

Wednesday, January 29, 2020

Proofs with $\vee$

Rules for justifications and subproofs

- When a rule calls for a subproof, we cite it as n–m, with first and last line numbers of the subproof.
- Assumptions **and** last line have to match rule.
- After a subproof is done, you can only cite the whole thing, and not any individual line in it.
- Subproofs can be nested.
- When that happens, you also can't cite any subproof entirely contained inside another subproof, once the surrounding subproof is done.

Reiteration

Which are correct applications of R?

1	$\quad A$	
2	$\quad\quad A$	
3	$\quad\quad A$	R 1
4	$\quad A$	R 1
5	$\quad A$	R 2
6	A	R 2
7	A	R 1

Reiteration

Which are correct applications of R?

1	A	
2	A	
3	A	✓ R 1
4	A	✓ R 1
5	A	✗ R 2
6	A	✗ R 2
7	A	✗ R 1

Introduction rule for $\vee$

We have $\mathscr{P} \vDash \mathscr{P} \vee \mathbb{Q}$. So:

m	$\mathscr{P}$	
	$\mathscr{P} \vee \mathbb{Q}$	$\vee$I m

m	$\mathbb{Q}$	
	$\mathscr{P} \vee \mathbb{Q}$	$\vee$I m

1	$\quad A$	
2	$\quad B \vee A$	$\vee$I 1
3	$A \to (B \vee A)$	$\to$I 1–2

Eliminating $\vee$

What can we justify with disjunction $\mathcal{P} \vee \mathcal{Q}$?

Not $\mathcal{P}$ and also not $\mathcal{Q}$: neither is entailed by $\mathcal{P} \vee \mathcal{Q}$.

But: if both $\mathcal{P}$ and $\mathcal{Q}$ separately entail some third sentence $\mathcal{R}$, then we know that $\mathcal{R}$ follows!

To show this, we need **two** proofs that show $\mathcal{R}$, but in each proof we are allowed to use one of $\mathcal{P}$, $\mathcal{Q}$.

Elimination rule for $\vee$

m	$\mathcal{P} \vee \mathcal{Q}$	
i	$\quad \mathcal{P}$	
	$\quad \vdots$	
j	$\quad \mathcal{R}$	
k	$\quad \mathcal{Q}$	
	$\quad \vdots$	
l	$\quad \mathcal{R}$	
	$\mathcal{R}$	$\vee$E m, i–j, k–l

1	$A \vee B$	
2	$\quad A$	
3	$\quad B \vee A$	$\vee$I 2
4	$\quad B$	
5	$\quad B \vee A$	$\vee$I 4
6	$B \vee A$	$\vee$E 1, 2–3, 4–5

1	$A \vee B$	
2	$A \to B$	
3	$\quad A$	
4	$\quad B$	$\to$E 2, 3
5	$\quad B$	
6	$\quad B$	R 5
7	B	$\vee$E 1, 3-4, 5-6

Lecture 9

Friday, January 31, 2020

Rules for $\bot$, $\neg$. Explosion and indirect proof

Contradictions

- In proofs, we don't just use the premises of the argument, but also sentences we've proved, and sentences we've assumed (for $\rightarrow$I, $\vee$E).
- Sometimes it happens that assumptions we must make for correct applications of these rules are incompatible with the premises.
- For instance, to prove the disjunctive syllogism $A \vee B, \neg B \models B$ using $\vee$E, the assumption B for the second case conflicts with the premise $\neg B$.

Disjunctive syllogism

1	$A \vee B$	
2	$\neg B$	
3	$\quad A$	
4	$\quad A$	R 3
5	$\quad B$	
	$\quad \vdots$	
k	$\quad A$	
$k+1$	A	$\vee$E 1, 3–4, 5–k

Contradictions: eliminating ¬

We highlight the situation where inside a subproof we have run into a contradictory situation by the symbol

$$\bot$$

m	$\neg\mathscr{P}$	
n	$\mathscr{P}$	
	$\bot$	$\neg$E m, n

Since this also eliminates a ¬, we'll call it ¬E.

Explosion

Any argument with jointly unsatisfiable premises is valid. So whenever we can justify $\perp$ in a proof, we should be able to justify **anything**. “From a contradiction, anything follows”

m	$\perp$	
k	$\mathscr{P}$	X m

Disjunctive syllogism

1	$A \lor B$	
2	$\neg B$	
3	$\quad A$	
4	$\quad A$	R 3
5	$\quad B$	
6	$\quad \bot$	$\neg$E 2, 5
7	$\quad A$	X 6
8	A	$\lor$E 1, 3–4, 5–7

Introducing $\neg$

- Recall from tutorial: An argument is valid iff the premises together with the negation of the conclusion are jointly unsatisfiable.
- For instance:
  - $\mathcal{Q} \therefore \mathcal{P}$ iff $\mathcal{Q}$ and $\neg\mathcal{P}$ are jointly unsatisfiable.
  - $\mathcal{Q} \therefore \neg\mathcal{P}$ iff $\mathcal{Q}$ and $\mathcal{P}$ are jointly unsatisfiable.
- This last one gives us idea for $\neg$I rule: To justify $\neg\mathcal{P}$, show that $\mathcal{P}$ (together with all other premises) is unsatisfiable.
- Unsatisfiable means: a contradiction ($\bot$) follows!

Introduction rule for $\neg$

$$
\begin{array}{l|l}
m & \begin{array}{|l} \mathscr{P} \\ \hline \vdots \\ \\ \bot \end{array} \\
n & \\
 & \neg\mathscr{P} \qquad \neg\mathrm{I}\ m\text{–}n
\end{array}
$$

1	$A \to B$	
2	$\neg B$	
3	A	
4	B	$\to$E 1, 3
5	$\bot$	$\neg$E 2, 4
6	$\neg A$	$\neg$I 3-5
7	$\neg B \to \neg A$	$\to$I 2-6
8	$(A \to B) \to (\neg B \to \neg A)$	$\to$I 1-7

Indirect proof rule

$$
\begin{array}{l|l}
m & \begin{array}{|l} \neg\mathscr{P} \\ \hline \vdots \\ \\ \bot \end{array} \\
n & \\
 & \mathscr{P} \qquad \texttt{IP}\ m\text{–}n
\end{array}
$$

1	$\neg A \to \neg B$	
2	B	
3	$\neg A$	
4	$\neg B$	$\to$E 1, 3
5	$\bot$	$\neg$E 2, 4
6	A	IP 3-5
7	$B \to A$	$\to$I 2-6
8	$(\neg A \to \neg B) \to (B \to A)$	$\to$I 1-7

Lecture 10

Monday, February 03, 2020

Proof strategies and examples

Proofs in TFL: strategies and examples

The rules, one more time: $\wedge$

$$
\begin{array}{r|ll}
m & \mathcal{P} & \\
n & \mathcal{Q} & \\
 & \mathcal{P} \wedge \mathcal{Q} & \wedge\mathrm{I}\ m,\ n
\end{array}
$$

$$
\begin{array}{r|ll}
m & \mathcal{P} \wedge \mathcal{Q} & \\
 & \mathcal{P} & \wedge\mathrm{E}\ m
\end{array}
$$

$$
\begin{array}{r|ll}
m & \mathcal{P} \wedge \mathcal{Q} & \\
 & \mathcal{Q} & \wedge\mathrm{E}\ m
\end{array}
$$

The rules, one more time: $\rightarrow$

$$
\begin{array}{l|l}
m & \mathcal{P} \\ \cline{2-2}
 & \vdots \\
n & \mathcal{Q} \\ \hline
 & \mathcal{P} \rightarrow \mathcal{Q} \qquad \rightarrow\mathrm{I}\ m\text{–}n
\end{array}
\qquad\qquad
\begin{array}{l|ll}
m & \mathcal{P} \rightarrow \mathcal{Q} & \\
n & \mathcal{P} & \\
 & \mathcal{Q} & \rightarrow\mathrm{E}\ m,\ n
\end{array}
$$

The rules, one more time: $\vee$

$$
\begin{array}{r|ll}
m & \mathcal{P} \vee \mathcal{Q} & \\
i & \quad \begin{array}{|l} \mathcal{P} \\ \hline \vdots \\ \\ \mathcal{R} \end{array} & \\
j & & \\
k & \quad \begin{array}{|l} \mathcal{Q} \\ \hline \vdots \\ \\ \mathcal{R} \end{array} & \\
l & & \\
 & \mathcal{R} & \vee\mathrm{E}\ m,\ i\text{–}j,\ k\text{–}l
\end{array}
$$

$$
\begin{array}{r|ll}
m & \mathcal{P} & \\
 & \mathcal{P} \vee \mathcal{Q} & \vee\mathrm{I}\ m
\end{array}
$$

$$
\begin{array}{r|ll}
m & \mathcal{Q} & \\
 & \mathcal{P} \vee \mathcal{Q} & \vee\mathrm{I}\ m
\end{array}
$$

The rules, one more time: ¬

$$
\begin{array}{r|ll}
m & \neg\mathscr{P} & \\
n & \mathscr{P} & \\
 & \bot & \neg\mathrm{E}\ m,\ n
\end{array}
\qquad\qquad
\begin{array}{r|l}
m & \begin{array}{|l} \mathscr{P} \\ \hline \vdots \\ \\ \bot \end{array} \\
n & \\
 & \neg\mathscr{P} \qquad \neg\mathrm{I}\ m\text{–}n
\end{array}
$$

The rules, one more time: R, X, and IP

$$\begin{array}{r|ll} m & \mathscr{P} & \\ k & \mathscr{P} & \mathsf{R}\ m \end{array}$$

$$\begin{array}{r|ll} m & \bot & \\ k & \mathscr{P} & \mathsf{X}\ m \end{array}$$

$$\begin{array}{r|l} m & \begin{array}{|l} \neg\mathscr{P} \\ \hline \vdots \\ \\ \bot \end{array} \\ n & \\ & \mathscr{P} \qquad \mathsf{IP}\ m\text{–}n \end{array}$$

Working forward and backward

- **Working backward** from a conclusion (goal) means:
  - Find main connective of goal sentence
  - Match with conclusion of corresponding I rule
  - Write out (above the goal!) what you'd need to apply that rule
- **Working forward** from a premise, assumption, or already justified sentence means:
  - Find main connective of premise, assumption, or sentence
  - Match with top premise of corresponding E rule
  - Write out what else you need to apply the E rule (new goals)
  - If necessary, write out conclusion of the rule

Constructing a proof

- Write out premises at the top (if there are any)
- Write conclusion at bottom
- Work backward & forward from goals and premises/assumptions in this order:
  - Work backward using $\wedge$I, $\rightarrow$I, $\leftrightarrow$I, $\neg$I, or forward using $\vee$E
  - Work forward using $\wedge$E
  - Work forward from $\neg$E if your goal is $\perp$
  - Work forward using $\rightarrow$E, $\leftrightarrow$E
  - Work backward from $\vee$I
  - Try IP
- Repeat for each new goal from top

1	$\neg(A \lor B)$	
2	$\quad A$	
3	$\quad A \lor B$	$\lor$I 2
4	$\quad \bot$	$\neg$E 1, 3
5	$\neg A$	$\neg$I 2-4
6	$\quad B$	
7	$\quad A \lor B$	$\lor$I 6
8	$\quad \bot$	$\neg$E 1, 7
9	$\neg B$	$\neg$I 6-8
10	$\neg A \land \neg B$	$\land$I 5, 9

1	$\neg A \land \neg B$	
2	$\quad A \lor B$	
3	$\quad\quad A$	
4	$\quad\quad \neg A$	$\land$E 1
5	$\quad\quad \bot$	$\neg$E 4, 3
6	$\quad\quad B$	
7	$\quad\quad \neg B$	$\land$E 1
8	$\quad\quad \bot$	$\neg$E 7, 6
9	$\quad \bot$	$\lor$E 2, 3-5, 6-8
10	$\neg(A \lor B)$	$\neg$I 2-9
11	$(\neg A \land \neg B) \to \neg(A \lor B)$	$\to$I 1-10

1	$\neg(A \vee \neg A)$	
2	A	
3	$A \vee \neg A$	$\vee$I 2
4	$\bot$	$\neg$E 1, 3
5	$\neg A$	$\neg$I 2–4
6	$A \vee \neg A$	$\vee$I 5
7	$\bot$	$\neg$E 1, 6
8	$A \vee \neg A$	IP 1–7

Lecture 11

Wednesday, February 05, 2020

Intro to FOL. Names and predicates. Connectives and ambiguity

Introduction to first-order logic
Chapter 21

Limits of symbolization in TFL

- Consider the argument:

  Greta is a hero.
  $\therefore$ There is a hero.

- It's clearly valid: in any case in which Greta is a hero, someone (or something, at least) is a hero, so there must be a hero.
- But its symbolization in TFL is invalid in TFL:

  G
  $\therefore$ H

The problem

- Symbolization in TFL allows us to break down sentences containing “and,” “or,” “if–then” and determine validity in virtue of such **connectives**.
- Anything that can’t be further broken down must be symbolized by a sentence letters.
- That includes basic sentences like “Greta is a hero,” but also:
  - Everyone is a hero.
  - No one is a hero.
  - All heroes wear capes.

The goals of FOL

- Finer-grained symbolization
- Combines with TFL
- Allows for precise semantics (like truth tables for TFL)
- Works with proof rules (add new rules)
- Be simple & expressive (few new symbols!)
- New language: **first-order logic FOL**

The goals of FOL

- Consider the valid argument:

  Greta is a hero.
  Greta does not wear a cape.
  $\therefore$ Not all heroes wear capes.

- We'll need way to connect the name "Greta" in the premises.
- We'll need to connect "hero" in the premise and conclusion.
- We want to retain use of $\neg$ for "not"
- We want the symbolization to have a proof.

First steps: names

- Purpose of a **proper name**: to pick out a single, specific thing.
- (Contrast with common nouns like “hero” or “rock” which pick out collections of things)
- For simplicity, we’ll only consider names that pick out an actual object (in any case we’re considering)
- Later on, we’ll be able to deal with other expressions that play a similar role to names, e.g., “the prime minister of Canada”
- In FOL, names will be lowercase letters a–r

First steps: predicates

- Take out a name from a sentence. What's left over is a **predicate**, e.g.,

  Greta is a hero

  $\Rightarrow$ $\_\_\_\_\_{}_x$ is a hero.

  Greta admires Autumn

  $\Rightarrow$ $\_\_\_\_\_{}_x$ admires $\_\_\_\_\_{}_y$.
- In FOL, predicates symbolized using uppercase letters A–Z plus a number of spaces marked with variables.
- Spaces correspond to blanks.

Symbolization keys

- Names: lowercase letters for proper names of English
- Predicates: uppercase letters with variables marking blanks.

$$
\begin{aligned}
a&: \text{Autumn}\\
g&: \text{Greta}\\
H(x)&: \underline{\qquad}_x \text{ is a hero}\\
I(x)&: \underline{\qquad}_x \text{ inspires}\\
C(x)&: \underline{\qquad}_x \text{ wears a cape}\\
W(x,y)&: \underline{\qquad}_x \text{ watched } \underline{\qquad}_y\\
A(x,y)&: \underline{\qquad}_x \text{ admires } \underline{\qquad}_y
\end{aligned}
$$

- Domain: what things we're talking about
  e.g., people alive in 2020

Symbolization

- Basic sentences:
  - Greta is a hero. $H(g)$
  - Greta admires Autum. $A(g, a)$
- Combinations using connectives:
  - Greta and Autumn are heroes. $H(g) \wedge H(a)$

Symbolization

- Replacing pronouns by antecedents:
  - If Autumn is a hero, Greta admires **her**. $H(a) \to A(g, a)$
  - Greta doesn't admire **herself**. $\neg A(g, g)$
  - Greta and Autumn watched **each other**. $W(g, a) \land W(a, g)$
- Modification:
  - Autumn is an **inspiring hero**.
    Autumn inspires and is a hero. $I(a) \land H(a)$
  - Greta is a **hero who doesn't wear a cape**.
    Greta is a hero and it's not the case that Greta wears a cape.
    $H(g) \land \neg C(g)$

Careful with modification

- Greta is an international hero.
  - Can't be paraphrased as "Greta is international and a hero"
  - So "_____ is an international hero" needs its own predicate
- The Piltdown Man is a fake fossil.
  - Can't be paraphrased as "The Piltdown Man is fake and a fossil"
  - "Fake" and other privative adjectives ("pretend," "fictitious") imply opposite!

Exercises

- Autumn and Greta are inspiring heroes.

Exercises

- Autumn and Greta are inspiring heroes.

  $(I(a) \wedge H(a)) \wedge (I(g) \wedge H(g))$

Exercises

- Autumn and Greta are inspiring heroes.

  $(I(a) \wedge H(a)) \wedge (I(g) \wedge H(g))$
- Greta admires Autumn but not herself.

Exercises

- Autumn and Greta are inspiring heroes.

  $(I(a) \wedge H(a)) \wedge (I(g) \wedge H(g))$
- Greta admires Autumn but not herself.

  $A(g, a) \wedge \neg A(g, g)$

Exercises

- Autumn and Greta are inspiring heroes.

  $(I(a) \wedge H(a)) \wedge (I(g) \wedge H(g))$
- Greta admires Autumn but not herself.

  $A(g, a) \wedge \neg A(g, g)$
- Greta inspires only if Autumn does.

Exercises

- Autumn and Greta are inspiring heroes.

  $(I(a) \wedge H(a)) \wedge (I(g) \wedge H(g))$
- Greta admires Autumn but not herself.

  $A(g, a) \wedge \neg A(g, g)$
- Greta inspires only if Autumn does.

  $I(g) \rightarrow I(a)$

Exercises

- Autumn and Greta are inspiring heroes.

  $(I(a) \wedge H(a)) \wedge (I(g) \wedge H(g))$
- Greta admires Autumn but not herself.

  $A(g, a) \wedge \neg A(g, g)$
- Greta inspires only if Autumn does.

  $I(g) \rightarrow I(a)$
- One of Greta and Autumn watched the other.

Exercises

- Autumn and Greta are inspiring heroes.

  $(I(a) \land H(a)) \land (I(g) \land H(g))$
- Greta admires Autumn but not herself.

  $A(g, a) \land \neg A(g, g)$
- Greta inspires only if Autumn does.

  $I(g) \to I(a)$
- One of Greta and Autumn watched the other.

  At least one:

  $W(g, a) \lor W(a, g)$

  Exactly one:

  $(W(g, a) \lor W(a, g)) \land \neg(W(g, a) \land W(a, g))$

Ambiguity

Types of ambiguity

- Lexical ambiguity: one word—many meanings
  e.g., “bank”, “crane”
- Syntactic ambiguity: one sentence—many readings
  e.g.,
  - “Flying planes can be dangerous” (Chomsky)
  - “One morning I shot an elephant in my pajamas.
    How he got in my pajamas, I don’t know.” (Groucho Marx)

Connectives and ambiguity

- a admires b and c or d
- a admires [[b and c] or d]
  $(A(a, b) \wedge A(a, c)) \vee A(a, d)$
- a admires [b and [c or d]]
  $A(a, b) \wedge (A(a, c) \vee A(a, d))$

The man who was hanged by a comma

- Sir Roger Casement (1864–1916)
- British consul to Congo and Peru
- Tried to recruit Irish revolutionaries in Germany during WWI
- Tried for treason

Treason Act of 1351

ITEM, Whereas divers Opinions have been before this Time in what Case Treason shall be said, and in what not; the King, at the Request of the Lords and of the Commons, hath made a Declaration in the Manner as hereafter followeth, that is to say; When a Man doth compass or imagine the Death of our Lord the King, or of our Lady his Queen or of their eldest Son and Heir; or if a Man do violate the King's Companion, or the King's eldest Daughter unmarried, or the Wife of the King's eldest Son and Heir; or **if a Man do levy War against our Lord the King in his Realm, or be adherent to the King's Enemies in his Realm, giving to them Aid and Comfort in the Realm or elsewhere**, and thereof be probably attainted of open Deed by the People of their Condition: . . . And it is to be understood, that in the Cases above rehearsed, that ought to be judged Treason which extends to our Lord the King, and his Royal Majesty: . . .

Treason Act of 1351

ITEM, Whereas divers Opinions have been before this Time in what Case Treason shall be said, and in what not; the King, at the Request of the Lords and of the Commons, hath made a Declaration in the Manner as hereafter followeth, that is to say; When a Man doth compass or imagine the Death of our Lord the King, or of our Lady his Queen or of their eldest Son and Heir; or if a Man do violate the King's Companion, or the King's eldest Daughter unmarried, or the Wife of the King's eldest Son and Heir; or **if a Man do levy War against our Lord the King in his Realm, or be adherent to the King's Enemies in his Realm, giving to them Aid and Comfort in the Realm, or elsewhere**, and thereof be probably attainted of open Deed by the People of their Condition: . . . And it is to be understood, that in the Cases above rehearsed, that ought to be judged Treason which extends to our Lord the King, and his Royal Majesty: . . .

R v. Casement in FOL

- c is adherent to the King's enemies in the realm, or gives the King's enemies aid and comfort in the realm[,] or elsewhere.

$$
\begin{aligned}
c&: \text{ Sir Roger Casement}\\
r&: \text{ the realm}\\
e&: \text{ elsewhere}\\
A(x,y)&: \ x \text{ is adherent to the King's enemies in } y\\
G(x,y)&: \ x \text{ gives aid and comfort to the King's enemies in } y
\end{aligned}
$$

- Without comma:

$$A(c,r) \vee (G(c,r) \vee G(c,e))$$

- With comma:

$$(A(c,r) \vee A(c,e)) \vee (G(c,r) \vee G(c,e))$$

Lecture 12

Friday, February 07, 2020

The existential quantifier

The existential quantifier
Chapter 22

Existential quantifier

- In English: “something,” “someone,” “there is . . . ”
- For instance:
  - Someone wears a cape.
  - There is a hero.
  - Something inspires.
- Note: can (often) go where names and pronouns also go.
- But works differently from names (“something” doesn’t pick out unique object).

How to symbolize “something”

- Idea: introduce a special term sg?
- Problem: can’t distinguish between
  - Someone is a hero and wears a cape.
  - Someone is a hero and someone wears a cape.

  as both would be symbolized by $H(sg) \wedge C(sg)$.
- Better idea: symbolize (complex) properties and introduce mechanism for expressing that properties are **instantiated**.

Expressing properties

- One–place predicates express properties, e.g.,
  - $H(x)$ expresses property "being a hero"
  - $I(x)$ expresses "is inspiring"
- Combinations of predicates (with connectives, names) can express derived properties, e.g.,
  - $A(x, g)$ expresses "admires Greta"
  - $W(a, x)$ expresses "is watched by Autumn"
  - $H(x) \wedge C(x)$ expresses "is a hero who wears a cape"
- Note: contains a single variable x

The existential quantifier $\exists$

- Symbol for "there is": $\exists$
- Combine $\exists$ with expression for a property (e.g., $(H(x) \wedge C(x))$) to say "something (someone) has that property"
- Put the variable that serves as a marker for the gap also after $\exists$. E.g.,

$$\exists x\,(H(x) \wedge C(x))$$

  says "Someone is a hero and wears a cape"

Quantifiers and variables

- Compare:

$$\exists x\,(H(x) \wedge C(x)) \text{ to}$$
$$\exists x\, H(x) \wedge \exists x\, C(x)$$

- In first case, the same person must be a hero and wear a cape.
- In second case, one person can be the hero, and another wears a cape.
- Note: Multiple $\exists x$ are independent, even if they use the same x. No difference in meaning between

$$\exists x\, H(x) \wedge \exists x\, C(x) \text{ and}$$
$$\exists x\, H(x) \wedge \exists y\, C(y).$$

The domain and quantifiers

- Symbolization key gives a domain of objects being talked about
- Quantifier **ranges over** this domain
- That means: $\exists x \ldots x \ldots$ is true iff some object **in the domain** has the property expressed by $\ldots x \ldots$
- Domain makes a difference: Consider $\exists x\, W(x, g)$
  - True if someone watched Greta (say, Autumn did).
  - Now take the domain to include only Greta.
  - Relative to that domain, $\exists x\, W(x, g)$ is true iff Greta watch herself.

Quantifier restriction

- “something” and “someone” work grammatically like singular terms (go where names can also go).
- “some” (on its own) does not: it is a **determiner** and needs a **complement**, e.g.,
  - a common noun (“some hero”), or
  - a noun phrase (“some admirer of Greta”).
- “some” + complement works like “someone”, e.g., “**Some hero** wears a cape”
- General form: “Some *F* is *G*.”

Quantifier restriction in FOL

- "Some F is G" **restricts** the "something" quantifier to Fs.
- We could (and linguists often do) mark restrictions in the quantifier, e.g., $(\exists x : F(x))G(x)$
- We won't because we can do without.
- "Some F is G" is true iff there is something which is both F and also G, so:
- "Some F is G" can be symbolized as

$$\exists x(F(x) \wedge G(x))$$

- We'll also symbolize the plural form this way ("Some Fs are Gs").
- And more generally (most) sentences of the form: "G(some F)" or "G(something that Fs)".

Examples

- Some hero wears a cape.
  Some heroes wear capes.

Examples

- Some hero wears a cape.
  Some heroes wear capes.
  $\exists x(H(x) \wedge C(x))$

Examples

- Some hero wears a cape.
  Some heroes wear capes.

  $\exists x(H(x) \wedge C(x))$
- Someone who wears a cape watched Greta.

Examples

- Some hero wears a cape.
  Some heroes wear capes.

  $\exists x(H(x) \wedge C(x))$
- Someone who wears a cape watched Greta.

  $\exists x(C(x) \wedge W(x, g))$

Examples

- Some hero wears a cape.
  Some heroes wear capes.

  $\exists x(H(x) \wedge C(x))$
- Someone who wears a cape watched Greta.

  $\exists x(C(x) \wedge W(x, g))$
- Greta admires someone who wears a cape.

Examples

- Some hero wears a cape.
  Some heroes wear capes.

  $\exists x(H(x) \wedge C(x))$
- Someone who wears a cape watched Greta.

  $\exists x(C(x) \wedge W(x, g))$
- Greta admires someone who wears a cape.

  $\exists x(C(x) \wedge A(g, x))$

Examples

- Some hero wears a cape.
  Some heroes wear capes.

  $\exists x(H(x) \wedge C(x))$
- Someone who wears a cape watched Greta.

  $\exists x(C(x) \wedge W(x, g))$
- Greta admires someone who wears a cape.

  $\exists x(C(x) \wedge A(g, x))$
- Autumn watched someone who watched Greta.

Examples

- Some hero wears a cape.
  Some heroes wear capes.

  $\exists x(H(x) \wedge C(x))$
- Someone who wears a cape watched Greta.

  $\exists x(C(x) \wedge W(x, g))$
- Greta admires someone who wears a cape.

  $\exists x(C(x) \wedge A(g, x))$
- Autumn watched someone who watched Greta.

  $\exists x(W(x, g) \wedge W(a, x))$

Lecture 13

Monday, February 10, 2020

The universal quantifier. More determiners: no, only, a.

The universal quantifier

The universal quantifier

- "Something is F" is true if at least one element of domain has F.
- "Everything is F" is true if every element of the domain has F.
- In FOL: $\forall x\, F(x)$.
- E.g.:
  - "Everyone wears a cape" $\forall x\, C(x)$
  - "Everyone watched Greta or Autumn" $\forall x (W(x, g) \vee W(x, a))$

All, every, any

- Determiners with universal meaning: all, every, any.
- Take complements like “some,” e.g.,
  - Every hero wears a cape.
  - All heroes inspire.
  - Any hero admires Greta.
- These are true in the same cases (mean the same).
- “Every F is G” is true iff everything which is F is G.

Restricted $\forall$ in FOL

- Suppose we can symbolize F and G.
- How do we symbolize "Every F is G"?
- Options:
  - $\forall x(F(x) \land G(x))$
  - $\forall x(F(x) \lor G(x))$
  - $\forall x(F(x) \to G(x))$

Restricted $\forall$ in FOL

- Suppose we can symbolize F and G.
- How do we symbolize "Every F is G"?
- Options:
  - $\forall x(F(x) \land G(x))$
    If true, everything must be F.
    So can be false when "Every F is G" is true.
  - $\forall x(F(x) \lor G(x))$
    True if everything is F (without being G).
    So can be true when "Every F is G" is false.
  - $\forall x(F(x) \to G(x))$
    If x is F, x must also be G.
    (If x is not F, doesn't matter if it's G or not.)

All Fs are Gs

Symbolize as $\forall x(F(x) \to G(x))$:

- All Fs are Gs.
- Every F is G.
- Any F is G.

Examples

- Every hero wears a cape.
  All heroes wear capes.

Examples

- Every hero wears a cape.
  All heroes wear capes.
  $\forall x(H(x) \to C(x))$

Examples

- Every hero wears a cape.
  All heroes wear capes.

  $\forall x(H(x) \to C(x))$
- Everyone who wears a cape watched Greta.

Examples

- Every hero wears a cape.
  All heroes wear capes.

  $\forall x(H(x) \to C(x))$
- Everyone who wears a cape watched Greta.

  $\forall x(C(x) \to W(x, g))$

Examples

- Every hero wears a cape.
  All heroes wear capes.

  $\forall x(H(x) \to C(x))$
- Everyone who wears a cape watched Greta.

  $\forall x(C(x) \to W(x, g))$
- Greta admires anyone who wears a cape.

Examples

- Every hero wears a cape.
  All heroes wear capes.

  $\forall x(H(x) \to C(x))$
- Everyone who wears a cape watched Greta.

  $\forall x(C(x) \to W(x, g))$
- Greta admires anyone who wears a cape.

  $\forall x(C(x) \to A(g, x))$

Examples

- Every hero wears a cape.
  All heroes wear capes.

  $\forall x(H(x) \to C(x))$
- Everyone who wears a cape watched Greta.

  $\forall x(C(x) \to W(x, g))$
- Greta admires anyone who wears a cape.

  $\forall x(C(x) \to A(g, x))$
- Autumn watched everyone who watched Greta.

Examples

- Every hero wears a cape.
  All heroes wear capes.

  $\forall x(H(x) \to C(x))$
- Everyone who wears a cape watched Greta.

  $\forall x(C(x) \to W(x, g))$
- Greta admires anyone who wears a cape.

  $\forall x(C(x) \to A(g, x))$
- Autumn watched everyone who watched Greta.

  $\forall x(W(x, g) \to W(a, x))$

No F is G

Symbolize "No Fs are Gs" as:

- $\forall x(F(x) \to \neg G(x))$
- $\neg\exists x(F(x) \land G(x))$

Examples

- No hero wears a cape.
  No heroes wear capes.

Examples

- No hero wears a cape.
  No heroes wear capes.
  $\forall x(H(x) \to \neg C(x))$

Examples

- No hero wears a cape.
  No heroes wear capes.

  $\forall x(H(x) \to \neg C(x))$
- No one who wears a cape watched Greta.

Examples

- No hero wears a cape.
  No heroes wear capes.

  $\forall x(H(x) \to \neg C(x))$
- No one who wears a cape watched Greta.

  $\forall x(C(x) \to \neg W(x, g))$

Examples

- No hero wears a cape.
  No heroes wear capes.

  $\forall x(H(x) \to \neg C(x))$
- No one who wears a cape watched Greta.

  $\forall x(C(x) \to \neg W(x, g))$
- Greta admires no one who wears a cape.

Examples

- No hero wears a cape.
  No heroes wear capes.

  $\forall x(H(x) \to \neg C(x))$
- No one who wears a cape watched Greta.

  $\forall x(C(x) \to \neg W(x, g))$
- Greta admires no one who wears a cape.

  $\neg\exists x(C(x) \land A(g, x))$

Examples

- No hero wears a cape.
  No heroes wear capes.

  $\forall x(H(x) \to \neg C(x))$
- No one who wears a cape watched Greta.

  $\forall x(C(x) \to \neg W(x, g))$
- Greta admires no one who wears a cape.

  $\neg\exists x(C(x) \land A(g, x))$
- Autumn watched no one who watched Greta.

Examples

- No hero wears a cape.
  No heroes wear capes.

  $\forall x(H(x) \to \neg C(x))$
- No one who wears a cape watched Greta.

  $\forall x(C(x) \to \neg W(x, g))$
- Greta admires no one who wears a cape.

  $\neg\exists x(C(x) \land A(g, x))$
- Autumn watched no one who watched Greta.

  $\forall x(W(x, g) \to \neg W(a, x))$

Only Fs are G

Symbolize "Only Fs are Gs" as:

$$\forall x(G(x) \to F(x))$$

Examples

- Only heroes wear capes.

Examples

- Only heroes wear capes.

  $\forall x(C(x) \to H(x))$

Examples

- Only heroes wear capes.

  $\forall x(C(x) \to H(x))$
- Only people who wear capes watched Greta.

Examples

- Only heroes wear capes.

  $\forall x(C(x) \to H(x))$

- Only people who wear capes watched Greta.

  $\forall x(W(x, g) \to C(x))$

Examples

- Only heroes wear capes.

  $\forall x(C(x) \to H(x))$
- Only people who wear capes watched Greta.

  $\forall x(W(x, g) \to C(x))$
- Greta admires only people who wear capes.

Examples

- Only heroes wear capes.

  $\forall x(C(x) \to H(x))$
- Only people who wear capes watched Greta.

  $\forall x(W(x, g) \to C(x))$
- Greta admires only people who wear capes.

  $\forall x(A(g, x) \to C(x))$

Examples

- Only heroes wear capes.

  $\forall x(C(x) \to H(x))$
- Only people who wear capes watched Greta.

  $\forall x(W(x, g) \to C(x))$
- Greta admires only people who wear capes.

  $\forall x(A(g, x) \to C(x))$
- Autumn watched only people who watched Greta.

Examples

- Only heroes wear capes.

  $\forall x(C(x) \to H(x))$
- Only people who wear capes watched Greta.

  $\forall x(W(x, g) \to C(x))$
- Greta admires only people who wear capes.

  $\forall x(A(g, x) \to C(x))$
- Autumn watched only people who watched Greta.

  $\forall x(W(a, x) \to W(x, g))$

The indefinite article

- We use “is a” to indicate predication, e.g., “Greta is a hero.”
- Often “a” is used to claim existence, e.g.,

  Greta admires a hero.

  $\exists x(H(x) \land A(g, x))$
- But a **generic** indefinite is closer to a universal quantifier:

  A hero is someone who inspires.

  $\forall x(H(x) \to I(x))$
- Be careful if the indefinite article is in the antecedent of a conditional:

  If **a hero** wears a cape, **they** inspire.

  $\forall x(H(x) \to I(x))$

Universal “Some”, existential “any”

- “Someone,” “something” can require a universal quantifier:
  In antecedent, with a pronoun referring back to it in consequent, e.g.,

  If **someone** is a hero, Autum admires **them**.
  Roughly: Autumn admires all heroes.
  $\forall x(H(x) \to A(a, x))$

- “Any” in antecedents but **without** pronouns referring back to them are existential:

  If anyone is a hero, Greta is.
  Roughly: if there are heroes (at all), Greta is a hero.
  $\exists x\, H(x) \to H(g)$

Lecture 14

Wednesday, February 12, 2020

Mixed domains. Expressing properties.

Mixed domains

- Sometimes you want to talk about more than one kind of thing.
- The domain can include any mix of things (e.g., people, animals, items of clothing)
- Proper symbolization then needs predicates for these kinds, e.g.:

Domain: people alive in 2020 and items of clothing

$P(x)$: $\_\_\_\_\__x$ is a person

$L(x)$: $\_\_\_\_\__x$ is an item of clothing.

$E(x)$: $\_\_\_\_\__x$ is a cape

$R(x, y)$: $\_\_\_\_\__x$ wears $\_\_\_\_\__y$

Quantification in mixed domains

- Not everyone is wearing a cape.

Quantification in mixed domains

- Not everyone is wearing a cape.
  - In domain of people only:

Quantification in mixed domains

- Not everyone is wearing a cape.
  - In domain of people only:
    $\neg\forall x\ C(x)$

Quantification in mixed domains

- Not everyone is wearing a cape.
  - In domain of people only:
    $\neg\forall x\ C(x)$
  - In mixed domain:

Quantification in mixed domains

- Not everyone is wearing a cape.
  - In domain of people only:
    $\neg\forall x\, C(x)$
  - In mixed domain:
    $\neg\forall x\, (P(x) \rightarrow C(x))$

Quantification in mixed domains

- Not everyone is wearing a cape.
  - In domain of people only:
    $\neg\forall x\, C(x)$
  - In mixed domain:
    $\neg\forall x\, (P(x) \rightarrow C(x))$
- Some people inspire.

Quantification in mixed domains

- Not everyone is wearing a cape.
  - In domain of people only:
    $\neg\forall x\, C(x)$
  - In mixed domain:
    $\neg\forall x\, (P(x) \rightarrow C(x))$
- Some people inspire.
  - In domain of people only:

Quantification in mixed domains

- Not everyone is wearing a cape.
  - In domain of people only:
    $\neg\forall x\, C(x)$
  - In mixed domain:
    $\neg\forall x\, (P(x) \rightarrow C(x))$
- Some people inspire.
  - In domain of people only:
    $\exists x\, I(x)$

Quantification in mixed domains

- Not everyone is wearing a cape.
  - In domain of people only:
    $\neg\forall x\, C(x)$
  - In mixed domain:
    $\neg\forall x\, (P(x) \rightarrow C(x))$
- Some people inspire.
  - In domain of people only:
    $\exists x\, I(x)$
  - In mixed domain:

Quantification in mixed domains

- Not everyone is wearing a cape.
  - In domain of people only:
    $\neg\forall x\, C(x)$
  - In mixed domain:
    $\neg\forall x\, (P(x) \rightarrow C(x))$
- Some people inspire.
  - In domain of people only:
    $\exists x\, I(x)$
  - In mixed domain:
    $\exists x (P(x) \wedge I(x))$

Quantification in mixed domains

- Not everyone is wearing a cape.
  - In domain of people only:
    $\neg\forall x\, C(x)$
  - In mixed domain:
    $\neg\forall x\, (P(x) \rightarrow C(x))$
- Some people inspire.
  - In domain of people only:
    $\exists x\, I(x)$
  - In mixed domain:
    $\exists x (P(x) \wedge I(x))$
- Greta wears something.

Quantification in mixed domains

- Not everyone is wearing a cape.
  - In domain of people only:
    $\neg\forall x\, C(x)$
  - In mixed domain:
    $\neg\forall x\, (P(x) \rightarrow C(x))$
- Some people inspire.
  - In domain of people only:
    $\exists x\, I(x)$
  - In mixed domain:
    $\exists x(P(x) \wedge I(x))$
- Greta wears something.

  $\exists x(L(x) \wedge R(g, x))$

Expressing properties, revisited

- One–place predicates express properties, e.g.,

  $H(x)$ expresses property "being a hero"
- Combinations of predicates (with connectives, names) can express derived properties, e.g.,

  $A(x, g)$ expresses "x admires Greta"
  $H(x) \wedge C(x)$ expresses "x is a hero who wears a cape"
- Using quantifiers, we can express even more complex properties, e.g.,

  $\exists y(P(y) \wedge A(x, y))$ expresses "x admires someone"

Finding, using properties expressed

- If you can say it for Greta, you can say it for x.
  - Greta admires a hero.
    $\exists y(H(y) \wedge A(g, y))$
  - x admires a hero.
    $\exists y(H(y) \wedge A(x, y))$
- If you can say it for x, you can say it for Greta.
  - x wears a cape.
    $\exists y(E(y) \wedge R(x, y))$
  - Greta wears a cape.
    $\exists y(E(y) \wedge R(g, y))$

$E(x)$: $\_\_\_\_\_{}_x$ is a cape
$R(x, y)$: $\_\_\_\_\_{}_x$ wears $\_\_\_\_\_{}_y$

Examples

- x wears a cape.

$P(x)$	$\_\_\_\_\__x$ is a person	$L(x)$	$\_\_\_\_\__x$ is an item of clothing
$E(x)$	$\_\_\_\_\__x$ is a cape	$R(x, y)$	$\_\_\_\_\__x$ wears $\_\_\_\_\__y$

Examples

- x wears a cape.
  $\exists y(E(y) \wedge R(x, y))$

$P(x)$	$\_\_\_\_\_{}_x$ is a person	$L(x)$	$\_\_\_\_\_{}_x$ is an item of clothing
$E(x)$	$\_\_\_\_\_{}_x$ is a cape	$R(x, y)$	$\_\_\_\_\_{}_x$ wears $\_\_\_\_\_{}_y$

Examples

- x wears a cape.
  $\exists y(E(y) \wedge R(x, y))$
- x is admired by everyone.

$P(x)$	$\_\_\_\_\_{}_x$ is a person	$L(x)$	$\_\_\_\_\_{}_x$ is an item of clothing
$E(x)$	$\_\_\_\_\_{}_x$ is a cape	$R(x, y)$	$\_\_\_\_\_{}_x$ wears $\_\_\_\_\_{}_y$

Examples

- x wears a cape.
  $\exists y(E(y) \wedge R(x, y))$
- x is admired by everyone.
  $\forall y(P(y) \rightarrow A(y, x))$

$P(x)$	$\_\_\_\_\_{}_x$ is a person	$L(x)$	$\_\_\_\_\_{}_x$ is an item of clothing
$E(x)$	$\_\_\_\_\_{}_x$ is a cape	$R(x, y)$	$\_\_\_\_\_{}_x$ wears $\_\_\_\_\_{}_y$

Examples

- x wears a cape.
  $\exists y(E(y) \wedge R(x, y))$
- x is admired by everyone.
  $\forall y(P(y) \rightarrow A(y, x))$
- x admires a hero.

$P(x)$	$\_\_\_\_\__x$ is a person	$L(x)$	$\_\_\_\_\__x$ is an item of clothing
$E(x)$	$\_\_\_\_\__x$ is a cape	$R(x, y)$	$\_\_\_\_\__x$ wears $\_\_\_\_\__y$

Examples

- x wears a cape.
  $\exists y(E(y) \wedge R(x, y))$
- x is admired by everyone.
  $\forall y(P(y) \rightarrow A(y, x))$
- x admires a hero.
  $\exists y(H(y) \wedge A(x, y))$

$P(x)$	$\_\_\_\_\_{}_x$ is a person	$L(x)$	$\_\_\_\_\_{}_x$ is an item of clothing
$E(x)$	$\_\_\_\_\_{}_x$ is a cape	$R(x, y)$	$\_\_\_\_\_{}_x$ wears $\_\_\_\_\_{}_y$

Examples

- x wears a cape.
  $\exists y(E(y) \wedge R(x, y))$
- x is admired by everyone.
  $\forall y(P(y) \rightarrow A(y, x))$
- x admires a hero.
  $\exists y(H(y) \wedge A(x, y))$
- x admires only heroes.

$P(x)$	$\_\_\_\_\_{}_x$ is a person	$L(x)$	$\_\_\_\_\_{}_x$ is an item of clothing
$E(x)$	$\_\_\_\_\_{}_x$ is a cape	$R(x, y)$	$\_\_\_\_\_{}_x$ wears $\_\_\_\_\_{}_y$

Examples

- x wears a cape.
  $\exists y(E(y) \wedge R(x, y))$
- x is admired by everyone.
  $\forall y(P(y) \rightarrow A(y, x))$
- x admires a hero.
  $\exists y(H(y) \wedge A(x, y))$
- x admires only heroes.
  $\forall y(A(x, y) \rightarrow H(y))$

$P(x)$	$\_\_\_\_\__x$ is a person	$L(x)$	$\_\_\_\_\__x$ is an item of clothing
$E(x)$	$\_\_\_\_\__x$ is a cape	$R(x, y)$	$\_\_\_\_\__x$ wears $\_\_\_\_\__y$

Examples

- x wears a cape.
  $\exists y(E(y) \wedge R(x, y))$
- x is admired by everyone.
  $\forall y(P(y) \rightarrow A(y, x))$
- x admires a hero.
  $\exists y(H(y) \wedge A(x, y))$
- x admires only heroes.
  $\forall y(A(x, y) \rightarrow H(y))$
- x is naked.

$P(x)$	$\_\_\_\_\__x$ is a person	$L(x)$	$\_\_\_\_\__x$ is an item of clothing
$E(x)$	$\_\_\_\_\__x$ is a cape	$R(x, y)$	$\_\_\_\_\__x$ wears $\_\_\_\_\__y$

Examples

- x wears a cape.
  $\exists y(E(y) \wedge R(x, y))$
- x is admired by everyone.
  $\forall y(P(y) \rightarrow A(y, x))$
- x admires a hero.
  $\exists y(H(y) \wedge A(x, y))$
- x admires only heroes.
  $\forall y(A(x, y) \rightarrow H(y))$
- x is naked.
  $\neg\exists y(L(y) \wedge R(x, y))$
  $\forall y(L(y) \rightarrow \neg R(x, y))$

$P(x)$	$\_\_\_\_\__x$ is a person	$L(x)$	$\_\_\_\_\__x$ is an item of clothing
$E(x)$	$\_\_\_\_\__x$ is a cape	$R(x, y)$	$\_\_\_\_\__x$ wears $\_\_\_\_\__y$

Lecture 15

Friday, February 14, 2020

Arguments and validity in FOL. Interpretations

Arguments and validity in FOL
Chapter 27

Arguments involving quantifiers

1. If an action x is morally wrong then A is blameworthy for freely doing x.
2. If x is rationally optimal (there is no action which A has reason to think there is more reason for A to do), then A is not blameworthy for freely doing x.
3. Therefore, if x is morally wrong, then x is not rationally optimal. (Principle of moral categoricity.)

(John Skorupski, *Ethical Explorations*, 2000 (link)

Symbolizing Skorupski

1. If an action x is morally wrong then A is blameworthy for freely doing x.

Domain:	people and actions
a:	an agent A
$W(x)$:	x is morally wrong
$B(x, y)$:	x is blameworthy for freely doing y

Symbolizing Skorupski

1. If an action x is morally wrong then A is blameworthy for freely doing x.

Domain:	people and actions
a:	an agent A
$W(x)$:	x is morally wrong
$B(x, y)$:	x is blameworthy for freely doing y

$$\forall x(W(x) \rightarrow B(a, x))$$

Symbolizing Skorupski

2. If x is rationally optimal, then A is not blameworthy for freely doing x.

Domain:	people and actions
a:	an agent A
$W(x)$:	x is morally wrong
$B(x, y)$:	x is blameworthy for freely doing y
$O(x)$:	x is rationally optimal

$$\forall x(W(x) \rightarrow B(a, x))$$

Symbolizing Skorupski

2. If x is rationally optimal, then A is not blameworthy for freely doing x.

Domain: people and actions
a: an agent A
$W(x)$: x is morally wrong
$B(x, y)$: x is blameworthy for freely doing y
$O(x)$: x is rationally optimal

$$\forall x(W(x) \rightarrow B(a, x))$$
$$\forall x(O(x) \rightarrow \neg B(a, x))$$

Symbolizing Skorupski

3. Therefore, if x is morally wrong, then x is not rationally optimal.

Domain:	people and actions
a:	an agent A
$W(x)$:	x is morally wrong
$B(x, y)$:	x is blameworthy for freely doing y
$O(x)$:	x is rationally optimal

$$\forall x(W(x) \rightarrow B(a, x))$$
$$\forall x(O(x) \rightarrow \neg B(a, x))$$

Symbolizing Skorupski

3. Therefore, if x is morally wrong, then x is not rationally optimal.

Domain: people and actions
a: an agent A
$W(x)$: x is morally wrong
$B(x, y)$: x is blameworthy for freely doing y
$O(x)$: x is rationally optimal

$$\forall x(W(x) \rightarrow B(a, x))$$
$$\forall x(O(x) \rightarrow \neg B(a, x))$$
$$\therefore \forall x(W(x) \rightarrow \neg O(x))$$

Symbolizing Skorupski

Domain: people and actions

a: an agent A

$W(x)$: x is morally wrong

$B(x, y)$: x is blameworthy for freely doing y

$O(x)$: x is rationally optimal

$$\begin{aligned} & \forall x(W(x) \to B(a, x)) \\ & \forall x(O(x) \to \neg B(a, x)) \\ \therefore\ & \forall x(W(x) \to \neg O(x)) \end{aligned}$$

All A are B

No C are B (iff No B are C)

∴ No A are C

Validity of arguments

Everyone is either a hero or a villain.
Not everyone is a villain.
Only villains are evil.
$\therefore$ Some heroes are good.

Validity in FOL

- Want to capture validity **in virtue of the meanings of the connectives and the quantifiers** (but ignoring meanings of predicate symbols)
- So we want to ignore any restrictions the predicate symbols place on their **extensions**
- Hence: allow **any** extension in a potential counterexample
- An argument is **first-order valid** if there is no **interpretation** in which the premises are true and the conclusion false

Forms of arguments

Everyone is either a hero or a villain.
Not everyone is a villain.
Only villains are evil.
$\therefore$ Some heroes are good.

$\forall x(H(x) \vee V(x))$
$\neg\forall x\, V(x)$
$\forall x(E(x) \rightarrow V(x))$
$\therefore \exists x(H(x) \wedge G(x))$

(In)validity of arguments

$$
\begin{aligned}
& \forall x(H(x) \vee V(x)) \\
& \neg\forall x\, V(x) \\
& \forall x(E(x) \rightarrow V(x)) \\
\therefore\ & \exists x(H(x) \wedge G(x))
\end{aligned}
$$

Domain: the inner planets

$H(x)$: x is smaller than Earth

$V(x)$: x has a moon

$E(x)$: x is inhabited

$G(x)$: x has rings

Interpretations

- Domain: collection of objects (not empty)
- **Referents** for each name (which object it names)
- Properties of each object
  - **Extension** of each 1–place predicate symbol: which objects it applies to
- Relations of each pair of objects
  - **Extension** of each 2–place predicate symbol: which pairs of objects standing in the relation

Extensions

Domain: the inner planets

$H(x)$: x is smaller than Earth

$V(x)$: x has a moon

$E(x)$: x is inhabited

$G(x)$: x has rings

Domain: Mercury, Venus, Earth, Mars

$H(x)$: Mercury, Venus, Mars

$V(x)$: Earth, Mars

$E(x)$: Earth

$G(x)$: —

(In)validity of arguments

$$
\begin{aligned}
& \forall x(H(x) \vee V(x)) \\
& \neg\forall x\, V(x) \\
& \forall x(E(x) \rightarrow V(x)) \\
\therefore\ & \exists x(H(x) \wedge G(x))
\end{aligned}
$$

Domain: Mercury, Venus, Earth, Mars
$H(x)$: Mercury, Venus, Mars
$V(x)$: Earth, Mars
$E(x)$: Earth
$G(x)$: —

(In)validity of arguments

$$
\begin{aligned}
& \forall x(H(x) \vee V(x)) \\
& \neg\forall x\, V(x) \\
& \forall x(E(x) \rightarrow V(x)) \\
\therefore\ & \exists x(H(x) \wedge G(x))
\end{aligned}
$$

Domain: 1, 2

$H(x)$: 1

$V(x)$: 2

$E(x)$: 2

$G(x)$: 2

(In)validity of arguments

$$\begin{aligned} & \forall x(H(x) \vee V(x)) \\ & \neg\forall x\, V(x) \\ & \forall x(E(x) \rightarrow V(x)) \\ \therefore\ & \exists x(H(x) \wedge G(x)) \end{aligned}$$

Domain: 1

$H(x)$:

$V(x)$:

$E(x)$:

$G(x)$:

(In)validity of arguments

$$\begin{aligned} &\forall x(H(x) \vee V(x)) \\ &\neg\forall x\, V(x) \\ &\forall x(E(x) \to V(x)) \\ \therefore\ &\exists x(H(x) \wedge G(x)) \end{aligned}$$

Domain: 1

$H(x)$: 1

$V(x)$: —

$E(x)$: —

$G(x)$: —

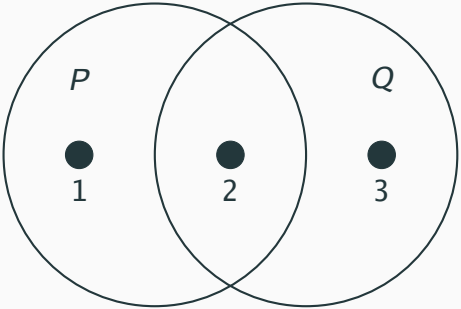
Extensions of predicates

Domain: 1, 2, 3

$P(x)$: 1, 2

$Q(x)$: 2, 3

$R(x)$: —



$R = \emptyset$

(In)validity of arguments

$$\begin{aligned} &\forall x(H(x) \vee V(x)) \\ &\neg\forall x\, V(x) \\ &\forall x(E(x) \to V(x)) \\ \therefore\ &\exists x(H(x) \wedge G(x)) \end{aligned}$$

Domain: 1, 2

$H(x)$: 1

$V(x)$: 2

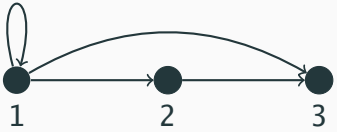
$E(x)$: 2

$G(x)$: 2

Extensions of predicates

Domain: 1, 2, 3

$A(x, y)$: $\langle 1, 1\rangle$, $\langle 1, 2\rangle$, $\langle 1, 3\rangle$, $\langle 2, 3\rangle$



Lecture 16

Monday, February 24, 2020

Truth of sentences in FOL. Using interpretations

Truth of sentences of FOL

- Given an interpretation I ...
- An **atomic sentence** is true iff the referents of the constants are in the extension of the predicate:
  - $P(a)$ is true iff referent r of a is in extension of P
  - $R(a, b)$ is true iff $\langle r, p \rangle$ is in extension of R
    (where r is referent of a, p is referent of b)
- $\neg\mathscr{A}$ is true iff $\mathscr{A}$ is false
- $\mathscr{A} \vee \mathscr{B}$ is true iff at least one of $\mathscr{A}$, $\mathscr{B}$ is true
- $\mathscr{A} \wedge \mathscr{B}$ is true iff both $\mathscr{A}$, $\mathscr{B}$ are true
- $\mathscr{A} \rightarrow \mathscr{B}$ is true iff $\mathscr{A}$ is false or $\mathscr{B}$ is true

Truth of quantified sentences

- $\exists x\, \mathcal{A}(x)$ is true iff $\mathcal{A}(x)$ is **satisfied** by **at least one** object in the domain
  - o satisfies $\mathcal{A}(x)$ iff $\mathcal{A}(c)$ is true in interpretation just like I, but with o as referent of c
- $\forall x\, \mathcal{A}(x)$ is true iff $\mathcal{A}(x)$ is **satisfied** by **every** object in the domain

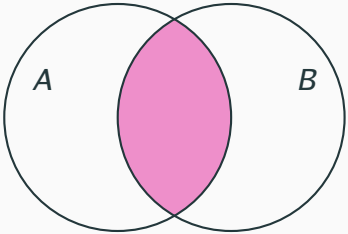
Truth of quantified sentences

- $\exists x\,(\mathscr{A}(x) \wedge \mathscr{B}(x))$ is true iff some object satisfies $\mathscr{A}(x) \wedge \mathscr{B}(x)$
  - o satisfies $\mathscr{A}(x) \wedge \mathscr{B}(x)$ iff it satisfies both $\mathscr{A}(x)$ and $\mathscr{B}(x)$
- $\forall x\,(\mathscr{A}(x) \rightarrow \mathscr{B}(x))$ is true iff every object satisfies $\mathscr{A}(x) \rightarrow \mathscr{B}(x)$
  - o satisfies $\mathscr{A}(x) \rightarrow \mathscr{B}(x)$ iff either
    - o does not satisfy $\mathscr{A}(x)$ or
    - o does satisfy $\mathscr{B}(x)$

Making “Some As are Bs” true

- $\exists x\,(A(x) \wedge B(x))$
- Extension of A and B must have something in common.

  (Filled area must contain at least one object)
- A and B can overlap, be equal, or be contained.
- Same situations make “No As are Bs” **false**.



Making “Some As are Bs” true

- $\exists x\,(A(x) \wedge B(x))$
- Extension of A and B must have something in common.

  (Filled area must contain at least one object)
- A and B can overlap, be equal, or be contained.
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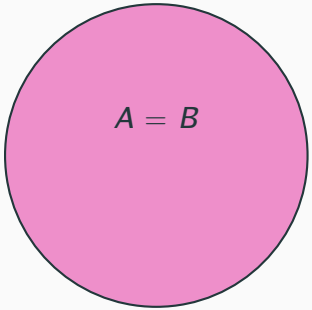

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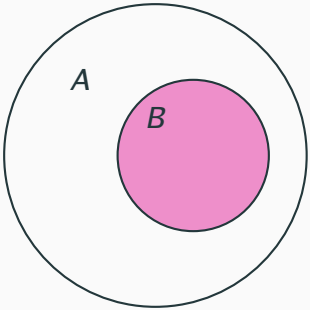

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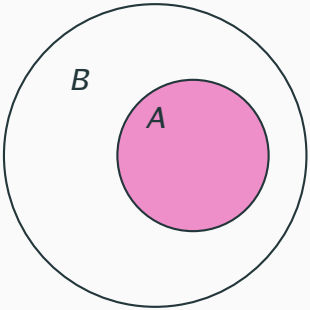
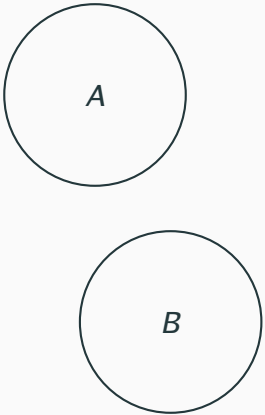


Making “Some As are Bs” false

- $\neg\exists x\,(A(x) \wedge B(x))$
- Extension of A and B must have nothing in common.
- A and B don’t overlap, or one or both is empty.
- Same situations make “No As are Bs” **true**.



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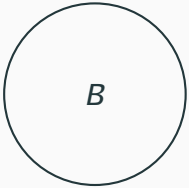

$A = \emptyset$

Making “Some As are Bs” false

- $\neg\exists x\,(A(x) \wedge B(x))$
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- A and B don’t overlap, or one or both is empty.
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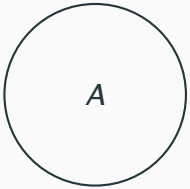
$B = \emptyset$



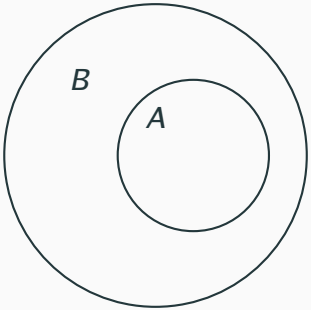
Making “Some As are Bs” false

- $\neg\exists x\,(A(x) \wedge B(x))$
- Extension of A and B must have nothing in common.
- A and B don’t overlap, or one or both is empty.
- Same situations make “No As are Bs” **true**.

$$A = B = \emptyset$$

Making “All As are Bs” true

- $\forall x\,(A(x) \to B(x))$
- Extension of A must be contained in extension of B.
- Extensions of A and B can be the same.
- Extension of A can be empty.
- Same situations make . . .
  - “Only Bs are As” **true**.
  - “Some As are not Bs” **false**.



Making "All As are Bs" true

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- Extension of A must be contained in extension of B.
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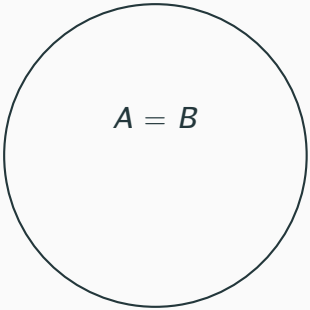

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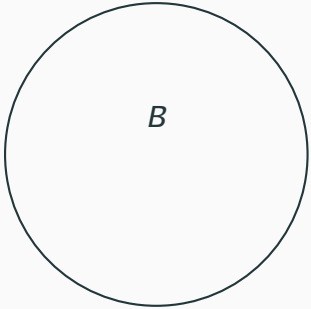
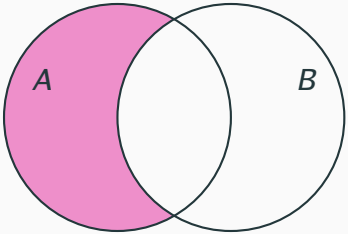


$A = \emptyset$

Making "All As are Bs" false

- $\forall x\,(A(x) \to B(x))$
- Extension of A must contain something not in B.
- Extensions of A cannot be empty, but B may be empty.
  - "Only Bs are As" **false**.
  - "Some As are not Bs" **true**.



Making "All As are Bs" false

- $\forall x\,(A(x) \to B(x))$
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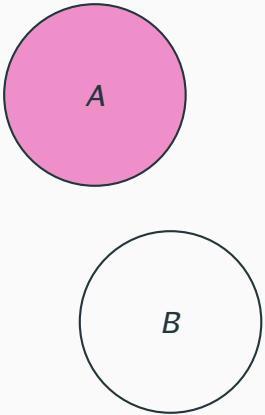

Making "All As are Bs" false

- $\forall x\,(A(x) \to B(x))$
- Extension of A must contain something not in B.
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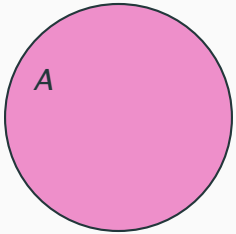

$B = \emptyset$

Lecture 17

Wednesday, February 26, 2020

Semantic notions of FOL

Semantic notions in FOL
Chapters 29–31

Semantics notions in FOL

- $\mathcal{P}_1, \ldots, \mathcal{P}_n \models \mathcal{Q}$ if no interpretation makes all of $\mathcal{P}_1, \ldots, \mathcal{P}_n$ true and $\mathcal{Q}$ false.
- $\mathcal{P}$ is a **validity** ($\models \mathcal{P}$) if it is true in every interpretation.
- $\mathcal{P}$ and $\mathcal{Q}$ are **equivalent in FOL** if no interpretation makes one true but the other false.
- $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are **jointly satisfiable in FOL** if some interpretation makes all of them true at the same time.

Using interpretations

- By providing one suitable interpretation we **can** show that...
  - an argument is **not valid** in FOL
  - a sentence is **not a validity** in FOL
  - two sentences are **not equivalent** in FOL
  - some sentences **are satisfiable** in FOL
- But we **cannot** show using any number of interpretations that...
  - an argument **is valid** in FOL
  - a sentence **is a validity** in FOL
  - two sentences **are equivalent** in FOL
  - some sentences **are not satisfiable** in FOL

Examples

- $\forall x(A(x) \vee B(x))$ and $\forall x\, A(x) \vee \forall x\, B(x)$ are not equivalent.
- $\forall x(A(x) \to B(x)), \forall x(A(x) \to \neg B(x))$ are jointly satisfiable.
- $\forall x(\neg A(x) \to B(x)), \exists x(B(x) \wedge C(x, b)) \not\models \exists x(\neg A(x) \wedge C(x, b))$.
- $\not\models \exists x\, A(a, x) \to \exists x\, A(x, x)$.

Test solutions on carnap.io

Arguing about Interpretations

- No interpretation(s) can show that an argument is valid.
- That's because there is no way to inspect all possible interpretations.
- But we can show that arguments are valid, by:
  - a formal proof (next time)
  - an informal argument
- The informal argument makes use of the **truth conditions** for sentences of FOL.
- Analogous to arguing about valuations in TFL.

Example

$$\forall x\, \mathscr{A}(x) \vee \forall x\, \mathscr{B}(x) \models \forall x(\mathscr{A}(x) \vee \mathscr{B}(x))$$

- Suppose an interpretation makes premise $\forall x\, \mathscr{A}(x) \vee \forall x\, \mathscr{B}(x)$ true.

Example

$$\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x) \models \forall x(\mathscr{A}(x) \lor \mathscr{B}(x))$$

- Suppose an interpretation makes premise $\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x)$ true.
- By the truth conditions for $\lor$, it makes either $\forall x\, \mathscr{A}(x)$ or $\forall x\, \mathscr{B}(x)$ true.

Example

$$\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x) \models \forall x(\mathscr{A}(x) \lor \mathscr{B}(x))$$

- Suppose an interpretation makes premise $\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x)$ true.
- By the truth conditions for $\lor$, it makes either $\forall x\, \mathscr{A}(x)$ or $\forall x\, \mathscr{B}(x)$ true.
- Suppose it's the first, i.e., $\forall x\, \mathscr{A}(x)$ is true.

Example

$$\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x) \models \forall x(\mathscr{A}(x) \lor \mathscr{B}(x))$$

- Suppose an interpretation makes premise $\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x)$ true.
- By the truth conditions for $\lor$, it makes either $\forall x\, \mathscr{A}(x)$ or $\forall x\, \mathscr{B}(x)$ true.
- Suppose it's the first, i.e., $\forall x\, \mathscr{A}(x)$ is true.
  - By the truth conditions for $\forall$, every object in the domain satisfies $\mathscr{A}(x)$.

Example

$$\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x) \models \forall x(\mathscr{A}(x) \lor \mathscr{B}(x))$$

- Suppose an interpretation makes premise $\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x)$ true.
- By the truth conditions for $\lor$, it makes either $\forall x\, \mathscr{A}(x)$ or $\forall x\, \mathscr{B}(x)$ true.
- Suppose it's the first, i.e., $\forall x\, \mathscr{A}(x)$ is true.
  - By the truth conditions for $\forall$, every object in the domain satisfies $\mathscr{A}(x)$.
  - By the truth conditions for $\lor$, every object satisfies $\mathscr{A}(x) \lor \mathscr{B}(x)$

Example

$$\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x) \models \forall x(\mathscr{A}(x) \lor \mathscr{B}(x))$$

- Suppose an interpretation makes premise $\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x)$ true.
- By the truth conditions for $\lor$, it makes either $\forall x\, \mathscr{A}(x)$ or $\forall x\, \mathscr{B}(x)$ true.
- Suppose it's the first, i.e., $\forall x\, \mathscr{A}(x)$ is true.
  - By the truth conditions for $\forall$, every object in the domain satisfies $\mathscr{A}(x)$.
  - By the truth conditions for $\lor$, every object satisfies $\mathscr{A}(x) \lor \mathscr{B}(x)$
  - So, by the truth conditions for $\forall$, $\forall x(\mathscr{A}(x) \lor \mathscr{B}(x))$ is true.

Example

$$\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x) \models \forall x(\mathscr{A}(x) \lor \mathscr{B}(x))$$

- Suppose an interpretation makes premise $\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x)$ true.
- By the truth conditions for $\lor$, it makes either $\forall x\, \mathscr{A}(x)$ or $\forall x\, \mathscr{B}(x)$ true.
- Suppose it's the first, i.e., $\forall x\, \mathscr{A}(x)$ is true.
  - By the truth conditions for $\forall$, every object in the domain satisfies $\mathscr{A}(x)$.
  - By the truth conditions for $\lor$, every object satisfies $\mathscr{A}(x) \lor \mathscr{B}(x)$
  - So, by the truth conditions for $\forall$, $\forall x(\mathscr{A}(x) \lor \mathscr{B}(x))$ is true.
- Suppose it's the second, i.e., $\forall x\, \mathscr{B}(x)$ is true: Similarly.

Example

$$\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x) \models \forall x(\mathscr{A}(x) \lor \mathscr{B}(x))$$

- Suppose an interpretation makes premise $\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x)$ true.
- By the truth conditions for $\lor$, it makes either $\forall x\, \mathscr{A}(x)$ or $\forall x\, \mathscr{B}(x)$ true.
- Suppose it's the first, i.e., $\forall x\, \mathscr{A}(x)$ is true.
  - By the truth conditions for $\forall$, every object in the domain satisfies $\mathscr{A}(x)$.
  - By the truth conditions for $\lor$, every object satisfies $\mathscr{A}(x) \lor \mathscr{B}(x)$
  - So, by the truth conditions for $\forall$, $\forall x(\mathscr{A}(x) \lor \mathscr{B}(x))$ is true.
- Suppose it's the second, i.e., $\forall x\, \mathscr{B}(x)$ is true: Similarly.
- These are the only possibilities: the interpretation must make the conclusion also true.

Lecture 18

Friday, February 28, 2020

Proofs with universal quantifiers

Proofs in FOL
Chapters 32 & 33

Rules for formal proofs

- Need rules for $\forall$ and $\exists$ for formal proofs
- Formal proofs now more important, because no alternative (truth–table method)
- Intro and Elim rules should be
  - simple
  - elegant (not involve other connectives or quantifiers)
  - yield only valid arguments

Candidates for rules

- Only simple sentence close to $\forall x\, \mathscr{A}(x)$ is $\mathscr{A}(c)$
- Gives simple, elegant $\forall$E rule:

$$
\begin{array}{c|ll}
k & \forall x\, \mathscr{A}(x) & \\
 & \mathscr{A}(c) & \forall\text{E } k
\end{array}
$$

- This is a good rule: $\forall x\, \mathscr{A}(x) \models \mathscr{A}(c)$.

Candidates for rules

- Problem: corresponding “intro rule” isn’t valid:

$$\begin{array}{c|ll} k & \mathscr{A}(c) & \\ & \forall x\, \mathscr{A}(x) & \textbf{doesn't follow from } k \end{array}$$

- Diagnosis: the c in $\mathscr{A}(c)$ is a name for a **specific object** We need a name for an **arbitrary, unspecified object**.
- Then, if $\mathscr{A}(c)$ is true for whatever c could name, then $\mathscr{A}(x)$ is satisfied by **every** object.

Names for arbitrary objects

- When we give proofs of general claims, we often do use names for arbitrary objects (well, mathematicians do at least).

  All heroes admire Greta.
  Only people who wear capes admire Greta.
  $\therefore$ All heroes wear capes.

Names for arbitrary objects

- When we give proofs of general claims, we often do use names for arbitrary objects (well, mathematicians do at least).

  All heroes admire Greta.
  Only people who wear capes admire Greta.
  $\therefore$ All heroes wear capes.

  Proof: Let Carl be any hero. Since all heroes admire Greta, Carl admires Greta. Since only people who wear capes admire Greta, Carl is wears a cape. But “Carl” stands for **any** hero. So all heroes wear capes.

Universal generalization

k	$\mathscr{A}(c)$	
	$\forall x\, \mathscr{A}(x)$	$\forall$I k

- c is special: c must not appear in any premise or assumption of a subproof not already ended
- $\mathscr{A}(x)$ is obtained from $\mathscr{A}(x)$ by replacing **all** occurrences of c by x.

General conditional proof

Proving "All As are Bs"

k	$\quad\vert\ A(c)$	
l	$\quad\vert\ B(c)$	
$l+1$	$A(c) \to B(c)$	$\to$I k–l
$l+2$	$\forall x(A(x) \to B(x))$	$\forall$I $l+1$

Example

All heroes admire Greta.
Only people who wear capes admire Greta.
$\therefore$ All heroes wear capes.

$$\forall x(H(x) \to A(x, g))$$
$$\forall x(A(x, g) \to C(x))$$
$$\therefore \forall x(H(x) \to C(x))$$

Let's do it on carnap.io

Example

1	$\forall x(H(x) \to A(x, g))$	
2	$\forall x(A(x, g) \to C(x))$	
3	$\quad H(c)$	
4	$\quad H(c) \to A(c, g)$	$\forall$E 1
5	$\quad A(c, g)$	$\to$E 4, 3
6	$\quad A(c, g) \to C(c)$	$\forall$E 2
7	$\quad C(c)$	$\to$E 6, 5
8	$H(c) \to C(c)$	$\to$I 3–7
9	$\forall x(H(x) \to C(x))$	$\forall$I 8

Example

1	$\forall x\, A(x) \lor \forall x\, B(x)$	
2	$\quad \forall x\, A(x)$	
3	$\quad A(c)$	$\forall$E 2
4	$\quad A(c) \lor B(c)$	$\lor$I 3
5	$\quad \forall x\, B(x)$	
6	$\quad B(c)$	$\forall$E 5
7	$\quad A(c) \lor B(c)$	$\lor$I 6
8	$A(c) \lor B(c)$	$\lor$E 1, 2–4, 5–7
9	$\forall x(A(x) \lor B(x))$	$\forall$I 8

Lecture 19

Monday, March 02, 2020

Proofs with existential quantifiers

Rules for $\exists$

- If we know of a specific object that it satisfies $\mathscr{A}(x)$, we know that at least one object satisfies $\mathscr{A}(x)$.
- So this rule is valid:

$$
\begin{array}{r|ll}
k & \mathscr{A}(c) & \\
 & \exists x\, \mathscr{A}(x) & \exists\mathrm{I}\ k
\end{array}
$$

Arbitrary objects again

- Problem: corresponding "elim rule" isn't valid:

$$\begin{array}{c|ll} k & \exists x\, \mathcal{A}(x) & \\ & \mathcal{A}(c) & \textbf{doesn't follow from } k \end{array}$$

- If we know that $\exists x\, \mathcal{A}(x)$ is true, we know that **some** objects satisfy $\mathcal{A}(x)$, but not which ones.
- To use this information, we have to introduce a temporary name that stands for any one of the objects that satisfy $\mathcal{A}(x)$.

Reasoning from existential information

- To use $\exists x\, \mathscr{A}(x)$, pretend the x has a name c, and reason from $\mathscr{A}(c)$.
- This is what we'd do if we reason informally from existential information, e.g.,

  There are heroes who wear capes.
  Anyone who wears a cape admires Greta.
  $\therefore$ Some heroes admire Greta.

Reasoning from existential information

- To use $\exists x\, \mathscr{A}(x)$, pretend the x has a name c, and reason from $\mathscr{A}(c)$.
- This is what we'd do if we reason informally from existential information, e.g.,

  There are heroes who wear capes.
  Anyone who wears a cape admires Greta.
  $\therefore$ Some heroes admire Greta.

  Proof: We know there are heroes who wear capes. Let Cate be an arbitrary one of them. So Cate wears a cape. Anyone who wears a cape admires Greta, Cate admires Greta. Since Cate is a hero who admires Greta, some heroes admire Greta.

Existential elimination

- If
  - we know that some object satisfies $\mathscr{A}(x)$,
  - we assume for the time being that c is one of them (i.e., assume $\mathscr{A}(c)$), and
  - we can prove that $\mathscr{B}$ follows from this assumption,

  then $\mathscr{B}$ follows already from $\exists x\, \mathscr{A}(x)$.
- Rule for existential elimination:

k	$\exists x\, \mathscr{A}(x)$	
m	$\quad \mathscr{A}(c)$	
n	$\quad \mathscr{B}$	
	$\mathscr{B}$	$\exists$E k, m–n

- c is special: c must not appear outside subproof

Example

There are heroes who wear capes.
Anyone who wears a cape admires Greta.
$\therefore$ Some heroes admire Greta.

$\exists x(H(x) \wedge C(x))$
$\forall x(C(x) \rightarrow A(x, g)$
$\therefore \exists x(H(x) \wedge A(x, g))$

Example

1	$\exists x(H(x) \land C(x))$	
2	$\forall x(C(x) \to A(x, g)$	
3	$H(c) \land C(c)$	
4	$C(c)$	$\land$E 3
5	$C(c) \to A(c, g)$	$\forall$E 2
6	$A(c, g)$	$\to$E 4, 5
7	$H(c)$	$\land$E 3
8	$H(c) \land A(c, g)$	$\land$I 4, 7
9	$\exists x(H(x) \land A(x, g))$	$\exists$I 8

Lecture 20
Wednesday, March 04, 2020
Test 1

This is a test!

- If you think you'll leave early, sit near the aisle or in the front row.
- Do not disturb your colleagues when you leave.
- **Do not leave during the last 10 minutes.**
- Set your phone to silent and put it away.
- Don't open the test until I tell you to.
- Put your name and ID on the front page.
- Read the instructions.
- The only things out should be the test, writing implement, and hydration/snacks.
- No cheating!
- You have 50 minutes to complete the test.
- Good luck!

Lecture 21

Friday, March 06, 2020

Multiple quantifiers and relations. Alternating quantifiers.

Multiple quantifiers
Chapter 23

Formulas expressing relations

- A formula $\mathscr{A}(x)$ with one free variable expresses a **property**
- A formula $\mathscr{B}(x, y)$ with two free variables expresses a **relation**
- $\forall x \forall y\, \mathscr{B}(x, y)$ is a sentence; it's true iff
  any pair of objects α, β stand in the relation expressed by $\mathscr{B}(x, y)$
- $\exists x \exists y\, \mathscr{B}(x, y)$ is a sentence; it's true iff
  at least one pair of objects α, β stand in the relation expressed by $\mathscr{B}(x, y)$

Multiple uses of a single quantifier: $\forall$

- $A(x, y)$ ... x admires y.
- $\forall x \forall y\, A(x, y)$ ... for every pair $\langle \alpha, \beta \rangle$, α admires β.
- in other words: everyone admires everyone.
- NB: "every pair" includes pairs $\langle \alpha, \alpha \rangle$, i.e.,
- $\forall x \forall y\, A(x, y)$ only true if all pairs $\langle \alpha, \alpha \rangle$ satisfy $A(x, y)$.
- That means, everyone admires themselves, in addition to everyone else.
- So: $\forall x \forall y\, A(x, y)$ does not symbolize "everyone admires everyone **else**."

Multiple uses of single quantifier: $\exists$

- $\exists x \exists y\, A(x, y)$ ... for at least one pair $\langle \alpha, \beta \rangle$, α admires β.
- In other words: at least one person admires at least one person.
- NB: includes pairs $\langle \alpha, \alpha \rangle$, i.e.,
- $\exists x \exists y\, A(x, y)$ is already true if a single pair $\langle \alpha, \alpha \rangle$ satisfies $A(x, y)$.
- That means, we could just have one person admiring themselves.
- So: $\exists x \exists y\, A(x, y)$ does not symbolize "someone admires someone **else**."

Alternating quantifiers

1. $\forall x \exists y\, A(x, y)$

2. $\forall y \exists x\, A(x, y)$

3. $\exists x \forall y\, A(x, y)$

4. $\exists y \forall x\, A(x, y)$

Alternating quantifiers

1. $\forall x \exists y\, A(x, y)$
   Everyone admires someone
   (possibly themselves)
2. $\forall y \exists x\, A(x, y)$

3. $\exists x \forall y\, A(x, y)$

4. $\exists y \forall x\, A(x, y)$

Alternating quantifiers

1. $\forall x \exists y\, A(x, y)$
   Everyone admires someone
   (possibly themselves)
2. $\forall y \exists x\, A(x, y)$
   Everyone is admired by someone
   (not necessarily the same person)
3. $\exists x \forall y\, A(x, y)$

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Convergence vs. Uniform Convergence

- A function f **point–wise continuous** if

$$\forall \epsilon \forall x \forall y \exists \delta (|x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon)$$

- A function f **uniformly continuous** if

$$\forall \epsilon \exists \delta \forall x \forall y (|x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon)$$

Our symbolization key

Domain:	people alive in 2020 and items of clothing
a:	Autumn
g:	Greta
$P(x)$:	$\_\_\_\_\__x$ is a person
$L(x)$:	$\_\_\_\_\__x$ is an item of clothing.
$E(x)$:	$\_\_\_\_\__x$ is a cape
$R(x, y)$:	$\_\_\_\_\__x$ wears $\_\_\_\_\__y$
$H(x)$:	$\_\_\_\_\__x$ is a hero
$I(x)$:	$\_\_\_\_\__x$ inspires
$Y(x, y)$:	$\_\_\_\_\__x$ is younger than $\_\_\_\_\__y$
$A(x, y)$:	$\_\_\_\_\__x$ admires $\_\_\_\_\__y$
$O(x, y)$:	$\_\_\_\_\__x$ owns $\_\_\_\_\__y$

Symbolizing multiple determiners

- What if your sentence contains more than one determiner phrase?
- Deal with each determiner separately
- Think of determiner phrase as replaced with name or variable—result has one less determiner
- When you're down to one determiner, apply known methods for single quantifiers
- This results in formulas that express properties or relations, but themselves contain quantifiers

Example

- All heroes wear a cape
- All heroes satisfy "x wears a cape"

$$\forall x(H(x) \rightarrow \text{“}x \text{ wears a cape”})$$

- x wears a cape

$$\exists y(E(y) \wedge R(x, y))$$

- Together:

$$\forall x(H(x) \rightarrow \exists y(E(y) \wedge R(x, y)))$$

Determiner within determiner phrase

- All heroes who wear a cape admire Greta.
- All things that satisfy "x is a hero who wears a cape" admire Greta.

$$\forall x(\text{“x is a hero who wears a cape”} \rightarrow A(x, g))$$

- x is a hero who wears a cape

$$H(x) \wedge \exists y(E(y) \wedge R(x, y))$$

- Together:

$$\forall x((H(x) \wedge \exists y(E(y) \wedge R(x, y))) \rightarrow A(x, g))$$

Mary Astell, 1666–1731

- British political philosopher
- *Some Reflections upon Marriage* (1700)
- In preface to 3rd ed. 1706 reacts to William Nicholls' claim (in *The Duty of Inferiors towards their Superiors, in Five Practical Discourses* (London 1701), Discourse IV: The Duty of Wives to their Husbands), that women are naturally inferior to men.

Astell taking down Nicholls

’Tis true, thro’ Want of Learning, and of that Superior Genius which Men as Men lay claim to, she [the author] was ignorant of the *Natural Inferiority* of our Sex, which our Masters lay down as a Self-Evident and Fundamental Truth. She saw nothing in the Reason of Things, to make this either a Principle or a Conclusion, but much to the contrary; it being Sedition at least, if not Treason to assert it in this Reign.

Astell taking down Nicholls

For if by the Natural Superiority of their Sex, they mean that *every* Man is by Nature superior to *every* Woman, which is the obvious meaning, and that which must be stuck to if they would speak Sense, it wou'd be a Sin in *any* Woman to have Dominion over *any* Man, and the greatest Queen ought not to command but to obey her Footman, because no Municipal Laws can supersede or change the Law of Nature; . . .

Astell taking down Nicholls

If they mean that *some* Men are superior to *some* Women this is no great Discovery; had they turn'd the Tables they might have seen that *some* Women are Superior to *some* Men. Or had they been pleased to remember their Oaths of Allegiance and Supremacy, they might have known that *One* Women is superior to *All* the Men in these Nations, or else they have sworn to very little purpose. (Mary

Astell, *Reflections upon Marriage*, 1706 Preface, iii–iv, and Mary Astell, *Political Writings*, ed. Patricia Springborg, Cambridge University Press, 1996, 9–10)

Symbolizing Astell

- Some woman is superior to every man
- Some woman satisfies "x is superior to every man"

$$\exists x(W(x) \wedge \text{“}x \text{ is superior to every man”})$$

- x is superior to every man

$$\forall y(M(y) \rightarrow S(x, y))$$

- Together:

$$\exists x(W(x) \wedge \forall y(M(y) \rightarrow S(x, y))$$

Lecture 22

Monday, March 09, 2020

Quantifier scope ambiguity. Donkey sentences.

Formalizing Astell

- Some woman is superior to some man.

Formalizing Astell

- Some woman is superior to some man.

  $\exists x(W(x) \wedge \exists y(M(y) \wedge S(x, y)))$

Formalizing Astell

- Some woman is superior to some man.

  $\exists x(W(x) \wedge \exists y(M(y) \wedge S(x, y)))$
- Every woman is superior to every man.

Formalizing Astell

- Some woman is superior to some man.

  $\exists x(W(x) \wedge \exists y(M(y) \wedge S(x, y)))$
- Every woman is superior to every man.

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Formalizing Astell

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  $\forall x(W(x) \rightarrow \forall y(M(y) \rightarrow S(x, y)))$
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Formalizing Astell

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  $\exists x(W(x) \wedge \exists y(M(y) \wedge S(x, y)))$
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  $\forall x(W(x) \rightarrow \exists y(M(y) \wedge S(x, y)))$

Formalizing Astell

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  $\forall x(W(x) \rightarrow \forall y(M(y) \rightarrow S(x,y)))$
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- Some woman is superior to every man

  $\exists x(W(x) \wedge \forall y(M(y) \rightarrow S(x,y)))$

“Any”

- Any cape is worn by a hero.

“Any”

- Any cape is worn by a hero.

$$\forall x(E(x) \to \exists y(H(y) \land R(y, x)))$$

- No hero wears any cape.

- No hero wears every cape.

“Any”

- Any cape is worn by a hero.

$$\forall x(E(x) \to \exists y(H(y) \land R(y, x)))$$

- No hero wears any cape.

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- No hero wears every cape.

$$\forall x(H(x) \to \neg\forall y(E(y) \to R(x, y))$$

Quantifier scope ambiguity

More scope ambiguity

- Autumn and Greta admire Isra or Luisa.

More scope ambiguity

- Autumn and Greta admire Isra or Luisa.
- Autumn admires Isra or Luisa, and so does Greta.

$$(A(a, i) \vee A(a, l)) \wedge$$
$$(A(g, i) \vee A(g, l))$$

More scope ambiguity

- Autumn and Greta admire Isra or Luisa.
- Autumn admires Isra or Luisa, and so does Greta.

$$(A(a, i) \vee A(a, l)) \wedge$$
$$(A(g, i) \vee A(g, l))$$

- Autumn and Greta both admire Isra, or they both admire Luisa.

$$(A(a, i) \wedge A(g, i)) \vee$$
$$(A(a, l) \wedge A(g, l))$$

Negation and the quantifiers

- "All heroes don't inspire"
  - Denial of "all heroes inspire"
    ("Do all heroes inspire? No, all heroes don't inspire")

$$\neg\forall x(H(x) \to I(x))$$
$$\exists x(H(x) \land \neg I(x))$$

  - All heroes are: not inspiring, i.e.,
    No heroes inspire

$$\forall x(H(x) \to \neg I(x))$$
$$\neg\exists x(H(x) \land I(x))$$

Multiple quantifiers and ambiguity

- "All heroes wear a cape"
  - "A cape" in the scope of "all heroes", i.e.,
    "For every hero, there is a cape they wear"

$$\forall x(H(x) \to \exists y(E(y) \land R(x, y))$$

  - "All heroes" in scope of "a cape", i.e.,
    "There is a cape which every hero wears"

$$\exists y(E(y) \land \forall x(H(x) \to R(x, y))$$

- Compare the joke: "Every day, a tourist is mugged on the streets of New York. We will interview him tonight."

Donkey sentences

Happy farmers

"Every farmer who owns a donkey is happy"

- Step–by–step symbolization: "All As are Bs"

Happy farmers

"Every farmer who owns a donkey is happy"

- Step–by–step symbolization: "All As are Bs"
- x is a farmer who owns a donkey ...

$$F(x) \wedge \exists y(D(y) \wedge O(x, y))$$

Happy farmers

"Every farmer who owns a donkey is happy"

- Step–by–step symbolization: "All As are Bs"
- x is a farmer who owns a donkey ...

$$F(x) \wedge \exists y(D(y) \wedge O(x, y))$$

- Every farmer who owns a donkey is happy

$$\forall x((F(x) \wedge \exists y(D(y) \wedge O(x, y))) \rightarrow H(x))$$

Unhappy donkeys

"Every farmer who owns a donkey beats it"

- Step-by-step symbolization: "All As are Bs"

Unhappy donkeys

"Every farmer who owns a donkey beats it"

- Step–by–step symbolization: "All As are Bs"
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Unhappy donkeys

"Every farmer who owns a donkey beats it"

- Step–by–step symbolization: "All As are Bs"
- x is a farmer who owns a donkey ...

$$F(x) \wedge \exists y(D(y) \wedge O(x, y))$$

- Every farmer who owns a donkey beats it

$$\forall x((F(x) \wedge \exists y(D(y) \wedge O(x, y))) \rightarrow B(x, y))$$

Symbolizing donkey sentences

"Every farmer who owns a donkey beats it"

- When is it false that every farmer who owns a donkey beats it? If there's a farmer who owns a donkey but doesn't beat it. Deny that!

$$\neg\exists x(F(x) \land \exists y(D(y) \land O(x,y) \land \neg B(x,y)))$$

Symbolizing donkey sentences

"Every farmer who owns a donkey beats it"

- When is it false that every farmer who owns a donkey beats it? If there's a farmer who owns a donkey but doesn't beat it. Deny that!

$$\neg\exists x(F(x) \land \exists y(D(y) \land O(x,y) \land \neg B(x,y)))$$

- For every farmer and every donkey they own: the farmer beats the donkey

$$\forall x \forall y((F(x) \land (D(y) \land O(x,y))) \to B(x,y)))$$

Symbolizing donkey sentences

"Every farmer who owns a donkey beats it"

- When is it false that every farmer who owns a donkey beats it? If there's a farmer who owns a donkey but doesn't beat it. Deny that!

$$\neg\exists x(F(x) \land \exists y(D(y) \land O(x,y) \land \neg B(x,y)))$$

- For every farmer and every donkey they own: the farmer beats the donkey

$$\forall x \forall y((F(x) \land (D(y) \land O(x,y))) \to B(x,y)))$$

- Every farmer beats every donkey they own

$$\forall x(F(x) \to \forall y((D(y) \land O(x,y)) \to B(x,y)))$$

Lecture 23

Wednesday, March 11, 2020

Proofs with alternating quantifiers

Proofs with alternating quantifiers

Rules for $\forall$

$$
\begin{array}{r|ll}
m & \forall x \mathcal{A}(\ldots x \ldots x \ldots) & \\
 & \mathcal{A}(\ldots c \ldots c \ldots) & \forall\mathrm{E}\ m
\end{array}
$$

Replace all x by the same c throughout.

$$
\begin{array}{r|ll}
m & \mathcal{A}(\ldots c \ldots c \ldots) & \\
 & \forall x \mathcal{A}(\ldots x \ldots x \ldots) & \forall\mathrm{I}\ m
\end{array}
$$

c must not occur in any undischarged assumption

Replace all c by the same x throughout. (In other words: c must not occur in $\forall x\, \mathcal{A}(\ldots)$)

x must not occur in $\mathcal{A}(\ldots c \ldots c \ldots)$

Introducing $\exists$

m	$\mathscr{A}(\ldots c \ldots c \ldots)$	
	$\exists x \mathscr{A}(\ldots x \ldots c \ldots)$	$\exists$I m

Replace **one or more** c by x.

Pick a x not already in $\mathscr{A}(\ldots c \ldots c \ldots)$

Eliminating $\exists$

m	$\exists x \mathscr{A}(\ldots x \ldots x \ldots)$	
i	$\quad \mathscr{A}(\ldots c \ldots c \ldots)$	
j	$\quad \mathscr{B}$	
	$\mathscr{B}$	$\exists$E m, i–j

Replace all x by the same c throughout, and pick a **new** c.

c must not occur in any undischarged assumption, in $\exists x \mathscr{A}(\ldots x \ldots x \ldots)$, or in $\mathscr{B}$

Admirers and admired

Someone is admired by everyone
$\therefore$ Everyone admires someone

$$\begin{array}{|l} \exists y \forall x\, A(x, y) \\ \hline \forall x \exists y\, A(x, y) \end{array}$$

Let's do it on carnap.io

Admirers and admired

1	$\exists y \forall x\, A(x, y)$	
2	$\quad \forall x\, A(x, c)$	
3	$\quad A(d, c)$	$\forall$E 2
4	$\quad \exists y\, A(d, y)$	$\exists$I 3
5	$\quad \forall x \exists y\, A(x, y)$	$\forall$I 4
6	$\forall x \exists y\, A(x, y)$	$\exists$E 1, 2–5

All hail Queen Anne

Some woman is superior to every man.
$\therefore$ Every man is inferior to some woman.

$$\begin{array}{|l} \exists y(W(y) \land \forall x(M(x) \to S(y,x))) \\ \hline \forall x(M(x) \to \exists y(W(y) \land S(y,x))) \end{array}$$

1	$\exists y(W(y) \land \forall x(M(x) \to S(y,x)))$	
2	$\quad W(c) \land \forall x(M(x) \to S(c,x))$	
3	$\quad\quad M(d)$	
4	$\quad\quad \forall x(M(x) \to S(c,x))$	$\land$E 2
5	$\quad\quad M(d) \to S(c,d))$	
6	$\quad\quad S(c,d)$	
7	$\quad\quad W(c)$	$\land$E 2
8	$\quad\quad W(c) \land S(c,d)$	$\land$I 6, 7
9	$\quad\quad \exists y(W(y) \land S(y,x))$	$\exists$I 8
10	$\quad M(d) \to \exists y(W(y) \land S(y,d))$	
11	$\quad \forall x(M(x) \to \exists y(W(y) \land S(y,x)))$	$\forall$I 10
12	$\forall x(M(x) \to \exists y(W(y) \land S(y,x)))$	$\exists$E 1, 2-11

Converses are invalid

$$\forall x \exists y\, A(x, y) \not\models \exists y \forall x\, A(x, y)$$

$$\forall x (M(x) \to \exists y (W(y) \land S(y, x))) \not\models \exists y (W(y) \land \forall x (M(x) \to S(y, x)))$$

Let's do it on carnap.io

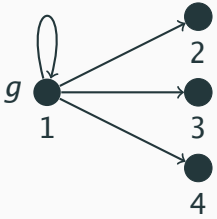
Lecture 24

Friday, March 13, 2020

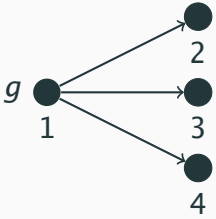
Identity. Uniqueness

Identity
Chapter 24

Greta admires everyone (else)

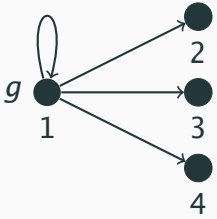


Greta admires everyone.



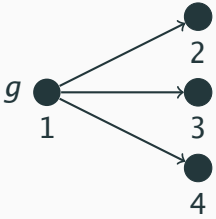
Greta admires everyone **else**.

Greta admires everyone (else)



Greta admires everyone.

$\forall x\, A(g, x)$



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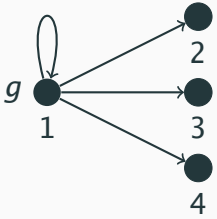

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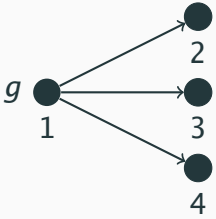
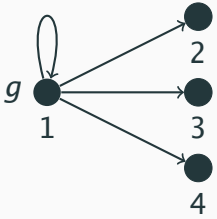


Greta admires everyone **else**.

$\forall x(\text{“}x \text{ is not Greta”} \rightarrow A(g, x))$

Greta admires everyone (else)



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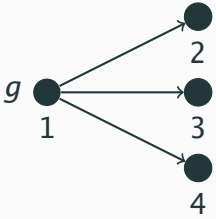

Greta admires everyone **else**.

$\forall x(\text{“}x \text{ is not Greta”} \to A(g, x))$

$\forall x(\neg x = g \to A(g, x))$

The identity predicate

- A new, special two-place predicate: $=$

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  - Written between arguments, without parentheses

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- $a = b$ true iff a and b are names for one and the the same object.

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- $a = b$ true iff a and b are names for one and the the same object.
- $x = y$ satisfied by all and only the pairs $\langle \alpha, \alpha \rangle$.

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- $\neg x = y$ is satisfied iff x and y are different objects.

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- $a = b$ true iff a and b are names for one and the the same object.
- $x = y$ satisfied by all and only the pairs $\langle \alpha, \alpha \rangle$.
- $\neg x = y$ is satisfied iff x and y are different objects.
- $x = \neg y$ is not grammatical. $\neg$ can only go in front of a formula, and y is not one.

Something else/everything else

- Remember: different variables does not mean different objects.

Something else/everything else

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- $\exists x \exists y\, A(x, y)$ doesn't mean that someone admires someone else.

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- To symbolize "someone else" add $\neg x = y$:

$$\exists x \exists y (\neg x = y \wedge A(x, y)$$

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- $\forall x \forall y\, A(x, y)$ says that everyone admires everyone including themselves.
- To symbolize "everyone else" add $\neg x = y$:

$$\forall x \forall y (\neg x = y \rightarrow A(x, y)$$

Something else/everything else

- The closest quantifier determines if you should use $\wedge$ or $\rightarrow$:

$$\forall x \exists y (\neg x = y \wedge A(x, y)) \qquad \exists x \forall y (\neg x = y \rightarrow A(x, y))$$

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- The closest quantifier determines if you should use $\wedge$ or $\rightarrow$:

$$\forall x \exists y (\neg x = y \wedge A(x, y)) \qquad \exists x \forall y (\neg x = y \rightarrow A(x, y))$$

- If you have mixed domains, it works the same way:

Something else/everything else

- The closest quantifier determines if you should use $\wedge$ or $\rightarrow$:

$$\forall x \exists y (\neg x = y \wedge A(x, y)) \qquad \exists x \forall y (\neg x = y \rightarrow A(x, y))$$

- If you have mixed domains, it works the same way:
- Everyone admires someone **else**:

$$\forall x (P(x) \rightarrow \exists y ((P(y) \wedge \neg x = y) \wedge A(x, y)))$$

Something else/everything else

- The closest quantifier determines if you should use $\wedge$ or $\rightarrow$:

$$\forall x \exists y (\neg x = y \wedge A(x, y)) \qquad \exists x \forall y (\neg x = y \rightarrow A(x, y))$$

- If you have mixed domains, it works the same way:
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$$\forall x (P(x) \rightarrow \exists y ((P(y) \wedge \neg x = y) \wedge A(x, y)))$$

- Someone admires everyone **else**:

$$\exists x (P(x) \wedge \forall y ((P(y) \wedge \neg x = y) \rightarrow A(x, y))$$

Singular “only”

- No–one other than Greta is a hero.

Singular “only”

- No–one other than Greta is a hero.

  $\neg\exists x(H(x) \wedge \neg x = g)$

Singular “only”

- No–one other than Greta is a hero.

  $\neg\exists x(H(x) \wedge \neg x = g)$

  $\forall x(H(x) \rightarrow x = g)$

Singular “only”

- No–one other than Greta is a hero.

  $\neg\exists x(H(x) \wedge \neg x = g)$

  $\forall x(H(x) \rightarrow x = g)$

- Only Greta is a hero.

Singular “only”

- No–one other than Greta is a hero.

  $\neg\exists x(H(x) \wedge \neg x = g)$

  $\forall x(H(x) \rightarrow x = g)$
- Only Greta is a hero.
- No–one other than Greta is a hero, and Greta is a hero.

Singular “only”

- No–one other than Greta is a hero.

  $\neg\exists x(H(x) \wedge \neg x = g)$

  $\forall x(H(x) \rightarrow x = g)$
- Only Greta is a hero.
- No–one other than Greta is a hero, and Greta is a hero.

  $\forall x(H(x) \rightarrow x = g) \wedge H(g)$

Singular “only”

- No–one other than Greta is a hero.

  $\neg\exists x(H(x) \wedge \neg x = g)$

  $\forall x(H(x) \rightarrow x = g)$
- Only Greta is a hero.
- No–one other than Greta is a hero, and Greta is a hero.

  $\forall x(H(x) \rightarrow x = g) \wedge H(g)$

  $\forall x(H(x) \leftrightarrow x = g))$

Uniqueness

- There is at least one hero.

$$\exists x\, H(x)$$

Uniqueness

- There is at least one hero.

$$\exists x\, H(x)$$

- There is exactly one hero.

Uniqueness

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  - There's at least one hero, and

Uniqueness

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  - There's at least one hero, and
  - There are no others

Uniqueness

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  - There's at least one hero, and
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$$\exists x\, (H(x) \wedge$$

Uniqueness

- There is at least one hero.

$$\exists x\, H(x)$$

- There is exactly one hero.
  - There's at least one hero, and
  - There are no others

$$\exists x\, (H(x) \land \neg \exists y\, (\neg y = x \land H(y)))$$

Uniqueness

- There is at least one hero.

$$\exists x\, H(x)$$

- There is exactly one hero.
  - There's at least one hero, and
  - There are no others

$$\exists x\, (H(x) \land \neg \exists y\, (\neg y = x \land H(y)))$$
$$\exists x\, (H(x) \land \forall y (H(y) \to x = y))$$

Uniqueness

- There is at least one hero.

$$\exists x\, H(x)$$

- There is exactly one hero.
  - There's at least one hero, and
  - There are no others

$$\exists x\, (H(x) \land \neg \exists y\, (\neg y = x \land H(y)))$$
$$\exists x\, (H(x) \land \forall y (H(y) \to x = y))$$

  - Or more succinctly: $\exists x \forall y (H(y) \leftrightarrow x = y)$

Lecture 25

Monday, March 16, 2020

Proofs with identity

Proofs with identity
Chapter 35

Everybody loves my baby

Brigitte Bardot singing “Everybody Loves my Baby”

Everybody loves my baby
My baby loves no–one but me
$\therefore$ My baby is me

$$\begin{array}{|l} \forall x\, L(x, b) \\ \forall x (L(b, x) \rightarrow x = i) \\ \hline b = i \end{array}$$

Let’s do it on carnap.io

Everybody loves my baby

1	$\forall x\, L(x, b)$	
2	$\forall x (L(b, x) \to x = i)$	
3	$L(b, b)$	$\forall$E 1
4	$L(b, b) \to b = i$	$\forall$E 2
5	$b = i$	$\to$E 3, 4

Proofs with identity

	$c = c$	=I
m	$a = b$	
n	$\mathcal{A}(\ldots a \ldots a \ldots)$	
	$\mathcal{A}(\ldots b \ldots a \ldots)$	=E m, n

m	$a = b$	
n	$\mathcal{A}(\ldots b \ldots b \ldots)$	
	$\mathcal{A}(\ldots a \ldots b \ldots)$	=E m, n

I am my baby

- We symbolized "My baby is me" as $b = i$.
- But it's equivalent to "I am my baby," $i = b$.
- $=$I and $=$E let us prove this:

1	$b = i$	
2	$b = b$	$=$I
3	$i = b$	$=$E 1, 2

Different properties, different things

- Two names d, e may name the same thing.

Different properties, different things

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- In that case, $d = e$ would be true.

Different properties, different things

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- And then anything that's true about d is also true about e.

Different properties, different things

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- In other words, $P(d), d = e \models P(e)$.

Different properties, different things

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- In other words, $P(d), d = e \models P(e)$.
- So if something is true about d but false about e, then $\neg d = e$.

Different properties, different things

- Two names d, e may name the same thing.
- In that case, $d = e$ would be true.
- And then anything that's true about d is also true about e.
- In other words, $P(d), d = e \models P(e)$.
- So if something is true about d but false about e, then $\neg d = e$.
- In other words:

$$P(d), \neg P(e) \models \neg d = e$$

Different properties, different things

1	$P(d)$	
2	$\neg P(e)$	
3	$\quad d = e$	
4	$\quad P(e)$	=E 1, 3
5	$\quad \bot$	$\neg$E 2, 4
6	$\neg d = e$	$\neg$I 2–5

Uniqueness, again

The two symbolizations of “there is exactly one hero” are equivalent:

$$\exists x(H(x) \wedge \forall y(H(y) \rightarrow x = y))$$
$$\exists x \forall y(H(y) \leftrightarrow x = y)$$

Uniqueness, again

1	$\exists x(H(x) \land \forall y(H(y) \to x = y))$	
2	$\quad H(a) \land \forall y(H(y) \to a = y)$	
3	$\quad\quad H(c)$	
4	$\quad\quad \forall y(H(y) \to a = y)$	$\land$E 2
5	$\quad\quad H(c) \to a = c$	$\forall$E 4
6	$\quad\quad a = c$	$\to$E 3, 5
7	$\quad\quad a = c$	
8	$\quad\quad H(a)$	$\land$E 2
9	$\quad\quad H(c)$	$=$E 7, 8
10	$\quad H(c) \leftrightarrow a = c$	$\leftrightarrow$I 3-6, 7-9
11	$\quad \forall y(H(y) \leftrightarrow a = y)$	$\forall$I 10
12	$\quad \exists x \forall y(H(y) \leftrightarrow x = y)$	$\exists$I 11
13	$\exists x \forall y(H(y) \leftrightarrow x = y)$	$\exists$E 1, 2-12

Uniqueness, again

1	$\exists x \forall y (H(y) \leftrightarrow x = y)$	
2	$\quad \forall y (H(y) \leftrightarrow a = y)$	
3	$\quad H(a) \leftrightarrow a = a$	$\forall$E 2
4	$\quad a = a$	$=$I
5	$\quad H(a)$	$\leftrightarrow$E 3, 4
6	$\quad\quad H(c)$	
7	$\quad\quad H(c) \leftrightarrow a = c$	$\forall$E 2
8	$\quad\quad a = c$	$\leftrightarrow$E 6, 7
9	$\quad H(c) \rightarrow a = c$	$\rightarrow$I 6-8
10	$\quad \forall y (H(y) \rightarrow a = y)$	$\forall$I 9
11	$\quad H(a) \wedge \forall y (H(y) \rightarrow a = y)$	$\wedge$I 5, 10
12	$\quad \exists x (H(x) \wedge \forall y (H(y) \rightarrow x = y))$	$\exists$I 11
13	$\exists x (H(x) \wedge \forall y (H(y) \rightarrow x = y))$	$\exists$E 1, 2-12

Lecture 26

Wednesday, March 18, 2020

Numerical quantification

Numerical quantification
Chapter 24

Numerical Quantification

- Cardinal numbers can be determiners:

Numerical Quantification

- Cardinal numbers can be determiners:
  - **Three heroes** wear capes.

Numerical Quantification

- Cardinal numbers can be determiners:
  - **Three heroes** wear capes.
- Not always clear if “three heroes” means **exactly** or **at least** three.

Numerical Quantification

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  - **Three heroes** wear capes.
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  - Do you have two dollars? Yes, I have two dollars. (Uncontroversially true even if you have more than 2$)

Numerical Quantification

- Cardinal numbers can be determiners:
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- FOL can express all of:

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- FOL can express all of:
  - **At least** n people are . . .

Numerical Quantification

- Cardinal numbers can be determiners:
  - **Three heroes** wear capes.
- Not always clear if “three heroes” means **exactly** or **at least** three.
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- FOL can express all of:
  - **At least** n people are ...
  - **Exactly** n people are ...

Numerical Quantification

- Cardinal numbers can be determiners:
  - **Three heroes** wear capes.
- Not always clear if "three heroes" means **exactly** or **at least** three.
- We'll assume the latter.
  - Do you have two dollars? Yes, I have two dollars. (Uncontroversially true even if you have more than 2$)
- FOL can express all of:
  - **At least** n people are ...
  - **Exactly** n people are ...
  - **At most** n people are ...

At least n

- At least 1 hero is inspiring:

$$\exists x(H(x) \wedge I(x))$$

At least n

- At least 1 hero is inspiring:

$$\exists x(H(x) \wedge I(x))$$

- At least 2 heroes are inspiring:

$$\exists x \exists y(\neg x = y \wedge ((H(x) \wedge I(x)) \wedge (H(y) \wedge I(y))))$$

At least n

- At least 1 hero is inspiring:

$$\exists x(H(x) \wedge I(x))$$

- At least 2 heroes are inspiring:

$$\exists x \exists y(\neg x = y \wedge ((H(x) \wedge I(x)) \wedge (H(y) \wedge I(y))))$$

- At least 3 heroes are inspiring:

$$\exists x \exists y \exists z((\neg x = y \wedge (\neg y = z \wedge \neg x = z)) \wedge \\ ((H(x) \wedge I(x)) \wedge ((H(y) \wedge I(y)) \wedge (H(z) \wedge I(z))))$$

At least n

- There are at least n As ("$\exists^{\geq n} x\, A(x)$"):

$$\exists x_1 \ldots \exists x_n$$

At least n

- There are at least n As ("$\exists^{\geq n} x\, A(x)$"):

$$\begin{aligned}\exists x_1 \ldots \exists x_n((\neg x_1 = x_2 \wedge (\neg x_1 = x_3 \wedge \cdots \wedge (\neg x_1 = x_n \wedge \\ (\neg x_2 = x_3 \wedge \cdots \wedge (\neg x_2 = x_n \wedge \\ \ddots \\ \neg x_{n-1} = x_n) \ldots) \wedge\end{aligned}$$

At least n

- There are at least n As ("$\exists^{\geq n} x\, A(x)$"):

$$\begin{aligned}
\exists x_1 \ldots \exists x_n ((\neg x_1 = x_2 \wedge (\neg x_1 = x_3 \wedge \cdots \wedge (\neg x_1 = x_n \wedge \\
(\neg x_2 = x_3 \wedge \cdots \wedge (\neg x_2 = x_n \wedge \\
\ddots \\
\neg x_{n-1} = x_n) \ldots) \wedge \\
(A(x_1) \wedge (A(x_2) \wedge \cdots \wedge A(x_n)) \ldots))
\end{aligned}$$

At least n

- Note: must state that **every pair** of variables is different, e.g.,

$$\exists x_1 \exists x_2 \exists x_3 ((\neg x_1 = x_2 \wedge \neg x_2 = x_3) \wedge (H(x_1) \wedge (H(x_2) \wedge H(x_3))))$$

  only says "There are at least two heroes"!

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  - Take extension of $H(x)$ to be: $1, 2$

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only says "There are at least two heroes"!

- Take extension of $H(x)$ to be: $1, 2$
- Then 1 can play role of x_1 and x_3, 2 role of x_2.

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only says "There are at least two heroes"!

- Take extension of $H(x)$ to be: $1, 2$
- Then 1 can play role of x_1 and x_3, 2 role of x_2.
- Both "$\neg 1 = 2$" and "$\neg 2 = 3$" are true.

At least n

- Note: must state that **every pair** of variables is different, e.g.,

$$\exists x_1 \exists x_2 \exists x_3 ((\neg x_1 = x_2 \wedge \neg x_2 = x_3) \wedge \\ (H(x_1) \wedge (H(x_2) \wedge H(x_3))))$$

  only says "There are at least two heroes"!
  - Take extension of $H(x)$ to be: $1, 2$
  - Then 1 can play role of x_1 and x_3, 2 role of x_2.
  - Both "$\neg 1 = 2$" and "$\neg 2 = 3$" are true.
- At least n Bs are Cs: take $B(x) \wedge C(x)$ for $A(x)$:

$$\exists^{\geq n} x (B(x) \wedge C(x))$$

Exactly one

- There is exactly one hero:

$$\exists x(H(x) \land \neg\exists y(H(y) \land \neg x = y))$$

Exactly one

- There is exactly one hero:

$$\exists x(H(x) \wedge \neg\exists y(H(y) \wedge \neg x = y))$$

- This is equivalent to:

$$\exists x(H(x) \wedge \forall y(H(y) \rightarrow x = y))$$

Exactly one

- There is exactly one hero:

$$\exists x(H(x) \wedge \neg\exists y(H(y) \wedge \neg x = y))$$

- This is equivalent to:

$$\exists x(H(x) \wedge \forall y(H(y) \rightarrow x = y))$$

- In general: "x has property A **uniquely**":

$$A(x) \wedge \forall y(A(y) \rightarrow x = y)$$

$$\text{or just:} \qquad \forall y(A(y) \leftrightarrow x = y)$$

Exactly n

- There are exactly n As ("$\exists^{=n} x\, A(x)$"):

$$\exists x_1 \ldots \exists x_n$$

Exactly n

- There are exactly n As ("$\exists^{=n} x\, A(x)$"):

$$\begin{aligned}\exists x_1 \ldots \exists x_n((\neg x_1 = x_2 \wedge (\neg x_1 = x_3 \wedge \cdots \wedge (\neg x_1 = x_n \wedge \\ (\neg x_2 = x_3 \wedge \cdots \wedge (\neg x_2 = x_n \wedge \\ \ddots \\ \neg x_{n-1} = x_n) \ldots)\end{aligned}$$

Exactly n

- There are exactly n As ("$\exists^{=n} x\, A(x)$"):

$$\begin{aligned}\exists x_1 \dots \exists x_n ((\neg x_1 = x_2 \wedge (\neg x_1 = x_3 \wedge \cdots \wedge (\neg x_1 = x_n \wedge \\ (\neg x_2 = x_3 \wedge \cdots \wedge (\neg x_2 = x_n \wedge \\ \ddots \\ \neg x_{n-1} = x_n) \dots) \wedge \\ (A(x_1) \wedge (A(x_2) \wedge \cdots \wedge A(x_n)) \dots))\end{aligned}$$

Exactly n

- There are exactly n As ("$\exists^{=n} x\, A(x)$"):

$$\begin{aligned}
\exists x_1 \ldots \exists x_n ((\neg x_1 = x_2 \wedge (\neg x_1 = x_3 \wedge \cdots \wedge (\neg x_1 = x_n \wedge \\
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\ddots \\
\neg x_{n-1} = x_n) \ldots) \wedge \\
(A(x_1) \wedge (A(x_2) \wedge \cdots \wedge A(x_n)) \ldots)) \wedge \\
\forall y (A(y) \rightarrow (y = x_1 \vee \cdots \vee y = x_n)))
\end{aligned}$$

Exactly n

- There are exactly n As ("$\exists^{=n} x\, A(x)$"):

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- Exactly n Bs are Cs:

$$\exists^{=n} x(B(x) \wedge C(x))$$

At most n

- There are **at most** n As $\Leftrightarrow$ There aren't **at least** $n+1$ As

$$\exists^{\leq n} x\, A(x) \Leftrightarrow \neg \exists^{\geq (n+1)} x\, A(x)$$

At most n

- There are **at most** n As $\Leftrightarrow$ There aren't **at least** $n+1$ As

$$\exists^{\leq n} x\, A(x) \Leftrightarrow \neg\exists^{\geq (n+1)} x\, A(x)$$

- For instance: There are at most two heroes:

$$\neg\exists x \exists y \exists z((H(x) \wedge (H(y) \wedge H(z))) \wedge (\neg x = y \wedge (\neg x = z \wedge \neg y = z)))$$

At most n

- There are **at most** n As $\Leftrightarrow$ There aren't **at least** $n+1$ As

$$\exists^{\leq n} x\, A(x) \Leftrightarrow \neg\exists^{\geq (n+1)} x\, A(x)$$

- For instance: There are at most two heroes:

$$\forall x \forall y \forall z ((H(x) \wedge (H(y) \wedge H(z))) \rightarrow (x = y \vee (x = z \vee y = z)))$$

At most n

- There are **at most** n As $\Leftrightarrow$ There aren't **at least** $n+1$ As

$$\exists^{\leq n} x\, A(x) \Leftrightarrow \neg\exists^{\geq (n+1)} x\, A(x)$$

- For instance: There are at most two heroes:

$$\forall x \forall y \forall z((H(x) \wedge (H(y) \wedge H(z))) \rightarrow (x = y \vee (x = z \vee y = z)))$$

- $\neg\exists^{\geq (n+1)} x\, A(x)$ is equivalent to:

$$\begin{aligned}
&\forall x_1 \ldots \forall x_{n+1}((A(x_1) \wedge \cdots \wedge A(x_{n+1})) \rightarrow \\
&(x_1 = x_2 \vee (x_1 = x_3 \vee \cdots \vee (x_1 = x_{n+1} \vee \\
&\qquad (x_2 = x_3 \vee \cdots \vee (x_2 = x_{n+1} \vee \\
&\qquad\qquad \ddots \\
&\qquad\qquad x_n = x_{n+1}) \ldots)))
\end{aligned}$$

Proofs with numerical claims

$$\begin{array}{|l}
\exists x\, P(x) \\
\forall x \forall y((P(x) \wedge P(y)) \to x = y) \\
\hline
\exists x(P(x) \wedge \forall y(P(y) \to x = y))
\end{array}$$

Proofs with numerical claims

1	$\exists x\, P(x)$	
2	$\forall x \forall y((P(x) \land P(y)) \to x = y)$	
3	$\quad P(a)$	
4	$\quad\quad P(c)$	
5	$\quad\quad \forall y((P(a) \land P(y)) \to a = y)$	$\forall$E 2
6	$\quad\quad (P(a) \land P(c)) \to a = c$	$\forall$E 5
7	$\quad\quad P(a) \land P(c)$	$\land$I 3, 4
8	$\quad\quad a = c$	$\to$E 6, 7
9	$\quad P(c) \to a = c$	$\to$I 4-8
10	$\quad \forall y(P(y) \to a = y)$	
11	$\quad P(a) \land \forall y(P(y) \to a = y)$	$\land$I 3, 11
12	$\quad \exists x(P(x) \land \forall y(P(y) \to x = y))$	$\exists$I 11
13	$\exists x(P(x) \land \forall y(P(y) \to x = y))$	$\exists$E 1, 3-12

Lecture 27

Friday, March 20, 2020

Definite descriptions: ‘the,’ ‘both,’ ‘neither’

“The”, “both”, “neither”
Chapter 25

Definite descriptions

- Definite description: **the so-and-so**

Definite descriptions

- Definite description: **the so-and-so**
- Russell's analysis of definite description: to say

  is to say:

Definite descriptions

- Definite description: **the so-and-so**
- Russell's analysis of definite description: to say
  - "The *A* is B"

  is to say:

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  is to say:
  - There is a unique *A*

Definite descriptions

- Definite description: **the so-and-so**
- Russell's analysis of definite description: to say
  - "The *A* is B"

  is to say:
  - There is a unique *A*
  - It is *B*

Definite descriptions

- Definite description: **the so-and-so**
- Russell's analysis of definite description: to say
  - "The A is B"

  is to say:
  - There is a unique A
  - It is B
- In FOL:

$$\exists x(A(x) \wedge \forall y(A(y) \rightarrow x = y) \wedge B(x))$$

Definite descriptions

- Definite description: **the so-and-so**
- Russell's analysis of definite description: to say
  - "The A is B"

  is to say:
  - There is a unique A
  - It is B
- In FOL:

$$\exists x(A(x) \wedge \forall y(A(y) \rightarrow x = y) \wedge B(x))$$

- or more succinctly:

$$\exists x(\forall y(A(y) \leftrightarrow x = y) \wedge B(x))$$

The vs. exactly one

- Compare:

The vs. exactly one

- Compare:
  1. The hero inspires:

$$\exists x(H(x) \wedge \forall y(H(y) \rightarrow x = y) \wedge I(x))$$

The vs. exactly one

- Compare:
  1. The hero inspires:

$$\exists x(H(x) \wedge \forall y(H(y) \rightarrow x = y) \wedge I(x))$$

  2. There is exactly one inspiring hero:

$$\exists x(H(x) \wedge \forall y((H(y) \wedge I(y)) \rightarrow x = y) \wedge I(x))$$

The vs. exactly one

- Compare:
  1. The hero inspires:

$$\exists x(H(x) \wedge \forall y(H(y) \rightarrow x = y) \wedge I(x))$$

  2. There is exactly one inspiring hero:

$$\exists x(H(x) \wedge \forall y((H(y) \wedge I(y)) \rightarrow x = y) \wedge I(x))$$

- (2) can be true without (1), but not vice versa.

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$$\exists x(H(x) \wedge \forall y((H(y) \wedge I(y)) \rightarrow x = y) \wedge I(x))$$

- (2) can be true without (1), but not vice versa.
- So (1) entails (2), but not vice versa.

Strawson's analysis

- According to Russell, "The hero wears a cape" is false if there is no hero, or if there is more than one.
- P. F. Strawson disagrees: we only succeed in making a statement if there is a unique hero.
- "There is a unique hero" is not part of what is **said**, but is only **presupposed**.

Singular possessive

- Singular possessives make noun phrases, e.g., “Deena’s cape”

Singular possessive

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Singular possessive

- Singular possessives make noun phrases, e.g., “Deena’s cape”
- They work like definite descriptions: Deena’s cape is the cape Deena owns. E.g.:
  - “Autumn wears Deena’s cape” symbolizes the same as: “Autumn wears the cape Deena owns”:

$$\begin{aligned}&\exists x[((E(x) \land O(d,x)) \land \\ &\quad \forall y((E(y) \land O(d,y)) \to x = y)) \land \\ &\quad W(a,x)]\end{aligned}$$

Singular vs. plural possessive

- Compare **plural** possessives: those are $\forall$'s:

Singular vs. plural possessive

- Compare **plural** possessives: those are $\forall$'s:
  - "Autumn wears Deena's cape**s**" symbolizes the same as:

Singular vs. plural possessive

- Compare **plural** possessives: those are $\forall$'s:
  - "Autumn wears Deena's cape**s**" symbolizes the same as: "Autumn wears every cape that Deena owns":

$$\forall x[(E(x) \land O(d, x)) \to W(a, x)]$$

Both

- “Both heroes inspire”: There are exactly 2 heroes, and both inspire:

$$\exists x \exists y$$

Both

- “Both heroes inspire”: There are exactly 2 heroes, and both inspire:

$$\exists x \exists y [((\neg x = y \wedge (H(x) \wedge H(y)))$$

Both

- "Both heroes inspire": There are exactly 2 heroes, and both inspire:

$$\exists x \exists y [((\neg x = y \wedge (H(x) \wedge H(y))) \wedge \\ \forall z (H(z) \rightarrow (z = x \vee z = y)))$$

Both

- “Both heroes inspire”: There are exactly 2 heroes, and both inspire:

$$\begin{aligned}\exists x \exists y [((\neg x = y \wedge (H(x) \wedge H(y))) \wedge \\ \forall z (H(z) \rightarrow (z = x \vee z = y))) \wedge \\ (I(x) \wedge I(y))]\end{aligned}$$

Both

- "Both heroes inspire": There are exactly 2 heroes, and both inspire:

$$\exists x \exists y [((\neg x = y \wedge (H(x) \wedge H(y))) \wedge \\ \forall z (H(z) \rightarrow (z = x \vee z = y))) \wedge \\ (I(x) \wedge I(y))]$$

- Note: "Both heroes inspire" implies "There are exactly two inspiring heroes", but not vice versa!

Neither

- "Neither hero is inspires": There are exactly 2 heroes, and neither of them inspires:

$$\begin{aligned}\exists x \exists y [((\neg x = y \wedge (H(x) \wedge H(y))) \wedge \\ \forall z (H(z) \rightarrow (z = x \vee z = y))) \wedge \\ (\neg I(x) \wedge \neg I(y))]\end{aligned}$$

Lecture 28

Monday, March 23, 2020

Implicature. Existential import

Existential import

- Does “all heroes wear capes” **entail** “there are heroes”?
- Not according to our symbolizations!

$$\forall x(H(x) \to C(x)) \not\models \exists x\, H(x)$$

- Why? If x is not a hero, $H(x) \to C(x)$ is true. So: if nothing is a hero, every x satisfies $H(x) \to C(x)$.
- (1) “Everyone who took the exam passed”
  (2) “Noone who took the exam failed”
  - (1) and (2) are equivalent
  - If noone took the exam, then (2) is true

Entailment

- P **entails** Q iff in no case where P is true, Q is false
- If P entails Q, then the denial of Q contradicts P (P, not Q are jointly impossible)
  - “Some A are B” entails “There are As”
  - “Some heroes wear capes” entails “There are heroes”
  - “Some heroes wear capes” and “There are no heroes” jointly impossible
- Does “All heroes wear capes” entail “There are heroes?”

Implicature

- P **implicates** Q if in asserting P, it is (strongly) suggested that Q is true
- Existential import is only **implicated**, not **entailed**
- Cancellation test: No contradiction if the implicature is denied:
  - "Noone who took the exam failed.
    In fact, noone took the exam at all"
  - "Some students passed the exam.
    In fact, all students passed."
  - "All unicorns are white.
    All zero of them."

Lecture 29
Wednesday, March 25, 2020
Test 2

This is a test!

- If you think you'll leave early, sit near the aisle or in the front row.
- Do not disturb your colleagues when you leave.
- **Do not leave during the last 10 minutes.**
- Set your phone to silent and put it away.
- Don't open the test until I tell you to.
- Put your name and ID on the front page.
- Read the instructions.
- The only things out should be the test, writing implement, and hydration/snacks.
- No cheating!
- You have 50 minutes to complete the test.
- Good luck!

Lecture 30

Friday, March 27, 2020

Expressive adequacy

Expressive adequacy and normal forms
Chapter 41

Truth Functions

Definition

An (n–place) **truth function** t is a mapping of n–tuples of **T** and **F** to either **T** or **F**.

n–place truth functions correspond to truth tables of sentence with n sentence letters $A_1, \ldots, A_n$.

		$t_\wedge$
T	T	T
T	F	F
F	T	F
F	F	F

$A_1 \wedge A_2$

		$t_\vee$
T	T	T
T	F	T
F	T	T
F	F	F

$A_1 \vee A_2$

Truth Functions

Definition

A sentence S containing the sentence letters $A_1, \ldots, A_n$ **expresses** the truth function t iff the truth value of S on the valuation which assigns v_i to A_i is $t(v_1, \ldots, v_n)$.

An n–place truth function is **expressible** if there is a sentence containing sentence letters $A_1, \ldots, A_n$ that expresses it.

Examples

		t_1
T	T	T
T	F	T
F	T	F
F	F	F

A_1 or: $A_1 \wedge (A_2 \vee \neg A_2)$

		t_{XOR}
T	T	F
T	F	T
F	T	T
F	F	F

$(A_1 \vee A_2) \wedge \neg(A_1 \wedge A_2)$

Expressive adequacy

Definition

A collection of connectives is **expressively adequate** if every truth function is expressible by a sentence containing only those connectives.

$\{\wedge, \vee\}$ not expressively adequate

- $\{\wedge, \vee\}$ is not expressively adequate
- Remember: To be expressively adequate, **every** truth function would have to be expressible using only $\wedge$ and $\vee$
- Which 2-place truth-functions can be expressed using $\wedge$ and $\vee$?
- Not this one:

T	T	F
T	F	T
F	T	T
F	F	F

Proof by induction

- Sometimes need to prove something for all sentences
- E.g., "every sentence containing only $\wedge$ and $\vee$ expresses a truth function **other than** t_{XOR}"
- Proof by induction:
  - Show that it holds for sentence letters
  - Suppose sentences $\mathcal{P}$, $\mathcal{Q}$ have the property and
  - Show that it then also holds for $(\mathcal{P} \wedge \mathcal{Q})$, $(\mathcal{P} \vee \mathcal{Q})$, etc.
- Why does this work? This is how we form sentences (involving only $\wedge$, $\vee$)

Proof by induction: example

- Claim: every sentence contains an even number of parentheses
- Proof strategy:
  - Show that every sentence letter contains an even number of parentheses
    - B, $\bot$
  - Show that if $\mathcal{P}$ contains an even number of parentheses, so does $\neg\mathcal{P}$
  - Show that if $\mathcal{P}$ and $\mathcal{Q}$ contain an even number of parentheses, so do

$$(\mathcal{P} \wedge \mathcal{Q}), (\mathcal{P} \vee \mathcal{Q}), (\mathcal{P} \rightarrow \mathcal{Q}), (\mathcal{P} \leftrightarrow \mathcal{Q})$$

$\wedge, \vee$ not expressively adequate

- Any sentence containing only A_1, A_2, $\wedge$, $\vee$ expresses a truth function t with $t(\mathbf{T}, \mathbf{T}) = \mathbf{T}$.
- Proof.
  - Sentence letters: A_1, A_2: express t_1, t_2.
  - Suppose $\mathcal{P}$, $\mathcal{Q}$ are sentences which contain only A_1, A_2, $\wedge$, $\vee$ and express truth functions t, t' with $t(\mathbf{T}, \mathbf{T}) = t'(\mathbf{T}, \mathbf{T}) = \mathbf{T}$
  - $(\mathcal{P} \wedge \mathcal{Q})$ expresses truth function s with
  $$s(\mathbf{T}, \mathbf{T}) = t_\wedge(t(\mathbf{T}, \mathbf{T}), t'(\mathbf{T}, \mathbf{T})) = \mathbf{T}$$
  - $(\mathcal{P} \vee \mathcal{Q})$ expresses truth function s with
  $$s(\mathbf{T}, \mathbf{T}) = t_\vee(t(\mathbf{T}, \mathbf{T}), t'(\mathbf{T}, \mathbf{T})) = \mathbf{T}$$
- Hence, no sentence containing only $\wedge$, $\vee$ can express a truth function t with $t(\mathbf{T}, \mathbf{T}) = \mathbf{F}$.

$\wedge, \vee, \neg$ is expressively adequate

A_1	A_2	A_3	t_{ODD}	S
T	T	T	T	
T	T	F	F	
T	F	T	F	
T	F	F	T	
F	T	T	F	
F	T	F	T	
F	F	T	T	
F	F	F	F	

$\wedge, \vee, \neg$ is expressively adequate

A_1	A_2	A_3	t_{ODD}	S
T	T	T	T	$(A_1 \wedge A_2 \wedge A_3)$
T	T	F	F	
T	F	T	F	
T	F	F	T	
F	T	T	F	
F	T	F	T	
F	F	T	T	
F	F	F	F	

$\wedge, \vee, \neg$ is expressively adequate

A_1	A_2	A_3	t_{ODD}	S
T	T	T	T	$(A_1 \wedge A_2 \wedge A_3)$
T	T	F	F	
T	F	T	F	
T	F	F	T	$(A_1 \wedge \neg A_2 \wedge \neg A_3)$
F	T	T	F	
F	T	F	T	
F	F	T	T	
F	F	F	F	

$\wedge, \vee, \neg$ is expressively adequate

A_1	A_2	A_3	t_{ODD}	S
T	T	T	T	$(A_1 \wedge A_2 \wedge A_3)$
T	T	F	F	
T	F	T	F	
T	F	F	T	$(A_1 \wedge \neg A_2 \wedge \neg A_3)$
F	T	T	F	
F	T	F	T	$(\neg A_1 \wedge A_2 \wedge \neg A_3)$
F	F	T	T	
F	F	F	F	

$\wedge, \vee, \neg$ is expressively adequate

A_1	A_2	A_3	t_{ODD}	S
T	T	T	T	$(A_1 \wedge A_2 \wedge A_3)$
T	T	F	F	
T	F	T	F	
T	F	F	T	$(A_1 \wedge \neg A_2 \wedge \neg A_3)$
F	T	T	F	
F	T	F	T	$(\neg A_1 \wedge A_2 \wedge \neg A_3)$
F	F	T	T	$(\neg A_1 \wedge \neg A_2 \wedge A_3)$
F	F	F	F	

$\wedge, \vee, \neg$ is expressively adequate

A_1	A_2	A_3	t_{ODD}	S
T	T	T	T	$(A_1 \wedge A_2 \wedge A_3) \vee$
T	T	F	F	
T	F	T	F	
T	F	F	T	$(A_1 \wedge \neg A_2 \wedge \neg A_3) \vee$
F	T	T	F	
F	T	F	T	$(\neg A_1 \wedge A_2 \wedge \neg A_3) \vee$
F	F	T	T	$(\neg A_1 \wedge \neg A_2 \wedge A_3)$
F	F	F	F	

$\wedge, \vee, \neg$ are expressively adequate

- Each line makes one, and only one, conjunction true, e.g.,
- $\neg A_1 \wedge A_2 \wedge \neg A_3$ is true in, and only in, line **F T F**
- Combine using $\vee$: make S true in all (and only) the lines where it is supposed to be true

The “neither...nor ...” connective: $\downarrow$

P	Q	$(P \downarrow Q)$
T	T	F
T	F	F
F	T	F
F	F	T

$\downarrow$ is expressively adequate

- Already know that $\{\neg, \wedge, \vee\}$ is truth-functionally complete, i.e.,
- Every truth-function can be expressed using only $\vee$, $\wedge$, $\neg$
- To show $\downarrow$ is expressively adequate, suffices to show that **every sentence containing only $\neg$, $\vee$, $\wedge$ is equivalent to one containing only $\downarrow$**
- For that, it suffices to show that any negated sentence, conjunction, disjunction, can be expressed using only $\downarrow$

Expressing $\neg$ using $\downarrow$

P	Q	$(P \downarrow Q)$
T	**T**	**F**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

- Note how $P \downarrow Q$ is **F** in the first line and **T** in the last (when P and Q have same truth value)
- So $P \downarrow P$ is **F** if P is **T**, and **T** if P is **F**, i.e.,

$$\neg P \Leftrightarrow (P \downarrow P)$$

Expressing $\vee$ using $\downarrow$

P	Q	$(P \downarrow Q)$
T	T	F
T	F	F
F	T	F
F	F	T

- $P \downarrow Q$ is the "neither . . . nor" connective, which can also be expressed as $\neg(P \vee Q)$, i.e.,

$$\neg(P \vee Q) \Leftrightarrow P \downarrow Q$$

Expressing $\vee$ using $\downarrow$

P	Q	$(P \downarrow Q)$
T	T	F
T	F	F
F	T	F
F	F	T

- $P \downarrow Q$ is the "neither . . . nor" connective, which can also be expressed as $\neg(P \vee Q)$, i.e.,

$$\neg(P \vee Q) \Leftrightarrow P \downarrow Q$$

- Negate both sides:

$$P \vee Q \Leftrightarrow \neg(P \downarrow Q)$$

Expressing $\vee$ using $\downarrow$

P	Q	$(P \downarrow Q)$
T	T	F
T	F	F
F	T	F
F	F	T

- $P \downarrow Q$ is the "neither . . . nor" connective, which can also be expressed as $\neg(P \vee Q)$, i.e.,

$$\neg(P \vee Q) \Leftrightarrow P \downarrow Q$$

- Negate both sides:

$$P \vee Q \Leftrightarrow \neg(P \downarrow Q)$$

- Apply what we figured out in last slide:

$$P \vee Q \Leftrightarrow (P \downarrow Q) \downarrow (P \downarrow Q)$$

Expressing $\wedge$ using $\downarrow$

- $P \downarrow Q$ is the "neither . . . nor" connective, which can also be expressed as $\neg P \wedge \neg Q$, i.e.,

$$(\neg P \wedge \neg Q) \Leftrightarrow P \downarrow Q$$

P	Q	$(P \downarrow Q)$
T	T	F
T	F	F
F	T	F
F	F	T

Expressing $\wedge$ using $\downarrow$

P	Q	$(P \downarrow Q)$
T	T	F
T	F	F
F	T	F
F	F	T

- $P \downarrow Q$ is the "neither ... nor" connective, which can also be expressed as $\neg P \wedge \neg Q$, i.e.,

$$(\neg P \wedge \neg Q) \Leftrightarrow P \downarrow Q$$

- Equivalence holds for **all sentences** P, Q, so also if we replace P by $\neg R$ and Q by $\neg S$:

$$\neg\neg R \wedge \neg\neg S \Leftrightarrow (\neg R \downarrow \neg S)$$

Expressing $\wedge$ using $\downarrow$

P	Q	$(P \downarrow Q)$
T	T	F
T	F	F
F	T	F
F	F	T

- $P \downarrow Q$ is the "neither ... nor" connective, which can also be expressed as $\neg P \wedge \neg Q$, i.e.,

$$(\neg P \wedge \neg Q) \Leftrightarrow P \downarrow Q$$

- Equivalence holds for **all sentences** P, Q, so also if we replace P by $\neg R$ and Q by $\neg S$:

$$\neg\neg R \wedge \neg\neg S \Leftrightarrow (\neg R \downarrow \neg S)$$

- Delete $\neg\neg$'s, and express $\neg$ using $\downarrow$:

$$R \wedge S \Leftrightarrow (R \downarrow R) \downarrow (S \downarrow S)$$

Expressively adequate connectives

- De Morgan's Law: $\wedge$ can be expressed by $\vee$ and $\neg$
- Similarly: $\vee$ can be expressed by $\wedge, \neg$
- So $\vee, \neg$ and $\wedge, \neg$ are expressively adequate
- $\rightarrow, \bot$ is expressively adequate (HW)
- $\rightarrow, \neg$ is expressively adequate
- No other sets of connectives that don't contain one of these sets are expressively adequate
- "Neither . . . nor" (NOR) is expressively adequate by itself
- "Not both" (NAND) connective is expressively adequate by itself
- No other 2–place connectives are expressively adequate by themselves

Lecture 31

Monday, March 30, 2020

Normal forms. Equivalence transformations

Disjunctive normal form

DNF

A sentence is in **disjunctive normal form** (DNF) if it

- contains only $\wedge$, $\vee$, $\neg$
- only sentence letters are in scope of $\neg$
- only sentence letters, $\wedge$, and $\neg$ are in scope of $\wedge$

In other words: DNF are disjunctions of conjunctions of sentence letters and negated sentence letters, e.g.:

$$(A \wedge \neg B) \vee ((\neg A \wedge C) \vee (B \wedge C))$$

$$\neg A \vee (B \wedge C)$$

$$A \wedge (B \wedge C)$$

Conjunctive normal form

CNF

A sentence is in **conjunctive normal form** (CNF) if it

- contains only $\wedge$, $\vee$, $\neg$
- only sentence letters are in scope of $\neg$
- only sentence letters, $\vee$, and $\neg$ are in scope of $\vee$

In other words: CNF are conjunctions of disjunctions of sentence letters and negated sentence letters, e.g.:

$$(A \vee \neg B) \wedge ((\neg A \vee C) \wedge (B \vee C))$$

$$\neg A \wedge (B \vee C)$$

$$A \vee (B \vee C)$$

DNF theorem

Theorem

Every sentence is equivalent to one in disjunctive normal form.

Proof: Construct truth table and apply method to show that the truth table can be expressed by a sentence involving only $\wedge$, $\vee$, $\neg$. That sentence is always in DNF.

CNF theorem

Theorem

Every sentence is equivalent to one in conjunctive normal form.

Proof: Construct truth table.

- For each line where sentence if **F**, write a disjunction of sentence letters and negated sentence letters:
  - Write A is A is assigned **F**.
  - Write $\neg A$ if A is assigned **T**
- Put $\wedge$'s between all of them.

The resulting sentence is true iff the original sentence is true, and is in CNF.

CNF from truth table

A_1	A_2	A_3	S	CNF
T	T	T	T	
T	T	F	F	$(\neg A_1 \vee (\neg A_2 \vee A_3)) \wedge$
T	F	T	F	$(\neg A_1 \vee (A_2 \vee \neg A_3)) \wedge$
T	F	F	T	
F	T	T	F	$(A_1 \vee (\neg A_2 \vee \neg A_3)) \wedge$
F	T	F	T	
F	F	T	T	
F	F	F	F	$(\neg A_1 \vee (\neg A_2 \vee \neg A_3))$

Transformation equivalences

Defining $\rightarrow$, $\leftrightarrow$

$$
\begin{aligned}
(\mathcal{P} \rightarrow \mathcal{Q}) &\Leftrightarrow (\neg\mathcal{P} \vee \mathcal{Q}) \\
(\mathcal{P} \leftrightarrow \mathcal{Q}) &\Leftrightarrow (\mathcal{P} \wedge \mathcal{Q}) \vee (\neg\mathcal{P} \wedge \neg\mathcal{Q}) \\
&\Leftrightarrow (\neg\mathcal{P} \vee \mathcal{Q}) \wedge (\mathcal{P} \vee \neg\mathcal{Q})
\end{aligned}
$$

Double negation

$$\neg\neg\mathcal{P} \Leftrightarrow \mathcal{P}$$

Transformation equivalences

De Morgan's Laws:

$$\neg(\mathcal{P} \vee \mathcal{Q}) \Leftrightarrow (\neg\mathcal{P} \wedge \neg\mathcal{Q})$$

$$\neg(\mathcal{P} \wedge \mathcal{Q}) \Leftrightarrow (\neg\mathcal{P} \vee \neg\mathcal{Q})$$

Commutativity:

$$\mathcal{P} \vee \mathcal{Q} \Leftrightarrow \mathcal{Q} \vee \mathcal{P}$$

$$\mathcal{P} \wedge \mathcal{Q} \Leftrightarrow \mathcal{Q} \wedge \mathcal{P}$$

Distributivity:

$$\mathcal{P} \vee (\mathcal{Q} \wedge \mathcal{R}) \Leftrightarrow (\mathcal{P} \vee \mathcal{Q}) \wedge (\mathcal{P} \vee \mathcal{R})$$

$$\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R}) \Leftrightarrow (\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{P} \wedge \mathcal{R})$$

Transforming sentences into CNF

- Replace any subsentence of the form $(\mathscr{P} \to \mathbb{Q})$, $(\mathscr{P} \leftrightarrow \mathbb{Q})$ by its equivalent.
- Use De Morgan's laws to place $\neg$'s in front of sentence letters
- Remove double negations.
- Use distributivity and commutativity to remove $\wedge$ in the scope of $\vee$.

Transforming sentences into CNF/DNF

- $\neg((A \leftrightarrow B) \wedge (B \rightarrow C))$

Transforming sentences into CNF/DNF

- $\neg((A \leftrightarrow B) \wedge (B \rightarrow C))$
- $\neg([(A \wedge B) \vee (\neg A \wedge \neg B)] \wedge (B \rightarrow C))$ Def$\leftrightarrow$

Transforming sentences into CNF/DNF

- $\neg((A \leftrightarrow B) \wedge (B \rightarrow C))$
- $\neg([(A \wedge B) \vee (\neg A \wedge \neg B)] \wedge (B \rightarrow C))$ Def$\leftrightarrow$
- $\neg([(A \wedge B) \vee (\neg A \wedge \neg B)] \wedge (\neg B \vee C))$ Def$\rightarrow$

Transforming sentences into CNF/DNF

- $\neg((A \leftrightarrow B) \wedge (B \rightarrow C))$
- $\neg([(A \wedge B) \vee (\neg A \wedge \neg B)] \wedge (B \rightarrow C))$ Def$\leftrightarrow$
- $\neg([(A \wedge B) \vee (\neg A \wedge \neg B)] \wedge (\neg B \vee C))$ Def$\rightarrow$
- $\neg[(A \wedge B) \vee (\neg A \wedge \neg B)] \vee \neg(\neg B \vee C)$ DeM

Transforming sentences into CNF/DNF

- $\neg((A \leftrightarrow B) \wedge (B \rightarrow C))$
- $\neg([(A \wedge B) \vee (\neg A \wedge \neg B)] \wedge (B \rightarrow C))$ Def$\leftrightarrow$
- $\neg([(A \wedge B) \vee (\neg A \wedge \neg B)] \wedge (\neg B \vee C))$ Def$\rightarrow$
- $\neg[(A \wedge B) \vee (\neg A \wedge \neg B)] \vee \neg(\neg B \vee C)$ DeM
- $[\neg(A \wedge B) \wedge \neg(\neg A \wedge \neg B)] \vee \neg(\neg B \vee C)$ DeM

Transforming sentences into CNF/DNF

- $\neg((A \leftrightarrow B) \wedge (B \rightarrow C))$
- $\neg([(A \wedge B) \vee (\neg A \wedge \neg B)] \wedge (B \rightarrow C))$ Def$\leftrightarrow$
- $\neg([(A \wedge B) \vee (\neg A \wedge \neg B)] \wedge (\neg B \vee C))$ Def$\rightarrow$
- $\neg[(A \wedge B) \vee (\neg A \wedge \neg B)] \vee \neg(\neg B \vee C)$ DeM
- $[\neg(A \wedge B) \wedge \neg(\neg A \wedge \neg B)] \vee \neg(\neg B \vee C)$ DeM
- $[(\neg A \vee \neg B) \wedge \neg(\neg A \wedge \neg B)] \vee \neg(\neg B \vee C)$ DeM

Transforming sentences into CNF/DNF

- $\neg((A \leftrightarrow B) \wedge (B \rightarrow C))$
- $\neg([(A \wedge B) \vee (\neg A \wedge \neg B)] \wedge (B \rightarrow C))$ Def$\leftrightarrow$
- $\neg([(A \wedge B) \vee (\neg A \wedge \neg B)] \wedge (\neg B \vee C))$ Def$\rightarrow$
- $\neg[(A \wedge B) \vee (\neg A \wedge \neg B)] \vee \neg(\neg B \vee C)$ DeM
- $[\neg(A \wedge B) \wedge \neg(\neg A \wedge \neg B)] \vee \neg(\neg B \vee C)$ DeM
- $[(\neg A \vee \neg B) \wedge \neg(\neg A \wedge \neg B)] \vee \neg(\neg B \vee C)$ DeM
- $[(\neg A \vee \neg B) \wedge (\neg\neg A \vee \neg\neg B)] \vee \neg(\neg B \vee C)$ DeM

Transforming sentences into CNF/DNF

- $\neg((A \leftrightarrow B) \wedge (B \rightarrow C))$
- $\neg([(A \wedge B) \vee (\neg A \wedge \neg B)] \wedge (B \rightarrow C))$ Def$\leftrightarrow$
- $\neg([(A \wedge B) \vee (\neg A \wedge \neg B)] \wedge (\neg B \vee C))$ Def$\rightarrow$
- $\neg[(A \wedge B) \vee (\neg A \wedge \neg B)] \vee \neg(\neg B \vee C)$ DeM
- $[\neg(A \wedge B) \wedge \neg(\neg A \wedge \neg B)] \vee \neg(\neg B \vee C)$ DeM
- $[(\neg A \vee \neg B) \wedge \neg(\neg A \wedge \neg B)] \vee \neg(\neg B \vee C)$ DeM
- $[(\neg A \vee \neg B) \wedge (\neg\neg A \vee \neg\neg B)] \vee \neg(\neg B \vee C)$ DeM
- $[(\neg A \vee \neg B) \wedge (\neg\neg A \vee \neg\neg B)] \vee (\neg\neg B \wedge \neg C)$ DeM

Transforming sentences into CNF/DNF

- $\neg((A \leftrightarrow B) \wedge (B \rightarrow C))$
- $\neg([(A \wedge B) \vee (\neg A \wedge \neg B)] \wedge (B \rightarrow C))$ Def$\leftrightarrow$
- $\neg([(A \wedge B) \vee (\neg A \wedge \neg B)] \wedge (\neg B \vee C))$ Def$\rightarrow$
- $\neg[(A \wedge B) \vee (\neg A \wedge \neg B)] \vee \neg(\neg B \vee C)$ DeM
- $[\neg(A \wedge B) \wedge \neg(\neg A \wedge \neg B)] \vee \neg(\neg B \vee C)$ DeM
- $[(\neg A \vee \neg B) \wedge \neg(\neg A \wedge \neg B)] \vee \neg(\neg B \vee C)$ DeM
- $[(\neg A \vee \neg B) \wedge (\neg\neg A \vee \neg\neg B)] \vee \neg(\neg B \vee C)$ DeM
- $[(\neg A \vee \neg B) \wedge (\neg\neg A \vee \neg\neg B)] \vee (\neg\neg B \wedge \neg C)$ DeM
- $[(\neg A \vee \neg B) \wedge (A \vee B)] \vee (B \wedge \neg C)$ DN

Transforming sentences into CNF

- $\underbrace{[(\neg A \vee \neg B) \wedge (A \vee B)]}_{\mathscr{P}} \vee (\underbrace{B}_{\mathscr{Q}} \wedge \underbrace{\neg C}_{\mathscr{R}})$

Transforming sentences into CNF

- $\underbrace{[(\neg A \vee \neg B) \wedge (A \vee B)]}_{\mathscr{P}} \vee (\underbrace{B}_{\mathscr{Q}} \wedge \underbrace{\neg C}_{\mathscr{R}})$
- $(\overbrace{[(\neg A \vee \neg B) \wedge (A \vee B)]}^{\mathscr{P}} \vee \overbrace{B}^{\mathscr{Q}}) \wedge (\overbrace{[(\neg A \vee \neg B) \wedge (A \vee B)]}^{\mathscr{P}} \vee \overbrace{\neg C}^{\mathscr{R}})$ Dist

Transforming sentences into CNF

- $\underbrace{[(\neg A \vee \neg B) \wedge (A \vee B)]}_{\mathcal{P}} \vee (\underbrace{B}_{\mathcal{Q}} \wedge \underbrace{\neg C}_{\mathcal{R}})$
- $(\overbrace{[(\neg A \vee \neg B) \wedge (A \vee B)]}^{\mathcal{P}} \vee \overbrace{B}^{\mathcal{Q}}) \wedge (\overbrace{[(\neg A \vee \neg B) \wedge (A \vee B)]}^{\mathcal{P}} \vee \overbrace{\neg C}^{\mathcal{R}})$ Dist
- $(\underbrace{B}_{\mathcal{P}} \vee [\underbrace{(\neg A \vee \neg B)}_{\mathcal{Q}} \wedge \underbrace{(A \vee B)}_{\mathcal{R}}]) \wedge$
  $([(\neg A \vee \neg B) \wedge (A \vee B)] \vee \neg C)$ Com

Transforming sentences into CNF

- $\underbrace{[(\neg A \vee \neg B) \wedge (A \vee B)]}_{\mathcal{P}} \vee (\underbrace{B}_{\mathcal{Q}} \wedge \underbrace{\neg C}_{\mathcal{R}})$
- $(\overbrace{[(\neg A \vee \neg B) \wedge (A \vee B)]}^{\mathcal{P}} \vee \overbrace{B}^{\mathcal{Q}}) \wedge (\overbrace{[(\neg A \vee \neg B) \wedge (A \vee B)]}^{\mathcal{P}} \vee \overbrace{\neg C}^{\mathcal{R}})$ Dist
- $(\underbrace{B}_{\mathcal{P}} \vee [\underbrace{(\neg A \vee \neg B)}_{\mathcal{Q}} \wedge \underbrace{(A \vee B)}_{\mathcal{R}}]) \wedge$
  $([(\neg A \vee \neg B) \wedge (A \vee B)] \vee \neg C)$ Com
- $([\overbrace{B}^{\mathcal{P}} \vee \overbrace{(\neg A \vee \neg B)}^{\mathcal{Q}}] \wedge [\overbrace{B}^{\mathcal{P}} \vee \overbrace{(A \vee B)}^{\mathcal{R}}]) \wedge$
  $([(\neg A \vee \neg B) \wedge (A \vee B)] \vee \neg C)$ Dist

Transforming sentences into CNF

- $\underbrace{[(\neg A \vee \neg B) \wedge (A \vee B)]}_{\mathcal{P}} \vee (\underbrace{B}_{\mathcal{Q}} \wedge \underbrace{\neg C}_{\mathcal{R}})$
- $(\overbrace{[(\neg A \vee \neg B) \wedge (A \vee B)]}^{\mathcal{P}} \vee \overbrace{B}^{\mathcal{Q}}) \wedge (\overbrace{[(\neg A \vee \neg B) \wedge (A \vee B)]}^{\mathcal{P}} \vee \overbrace{\neg C}^{\mathcal{R}})$ Dist
- $(\underbrace{B}_{\mathcal{P}} \vee [\underbrace{(\neg A \vee \neg B)}_{\mathcal{Q}} \wedge \underbrace{(A \vee B)}_{\mathcal{R}}]) \wedge$
  $([(\neg A \vee \neg B) \wedge (A \vee B)] \vee \neg C)$ Com
- $([\overbrace{B}^{\mathcal{P}} \vee \overbrace{(\neg A \vee \neg B)}^{\mathcal{Q}}] \wedge [\overbrace{B}^{\mathcal{P}} \vee \overbrace{(A \vee B)}^{\mathcal{R}}]) \wedge$
  $([(\neg A \vee \neg B) \wedge (A \vee B)] \vee \neg C)$ Dist
- $([B \vee (\neg A \vee \neg B)] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Dist

Simplification equivalences

Associativity:

$$\mathcal{P} \vee (\mathcal{Q} \vee \mathcal{R}) \Leftrightarrow (\mathcal{P} \vee \mathcal{Q}) \vee \mathcal{R}$$

$$\mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{R}) \Leftrightarrow (\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{R}$$

Idempotence:

$$(\mathcal{P} \vee \mathcal{P}) \Leftrightarrow \mathcal{P} \qquad (\mathcal{P} \wedge \mathcal{P}) \Leftrightarrow \mathcal{P}$$

Absorption:

$$\mathcal{P} \wedge (\mathcal{P} \vee \mathcal{Q}) \Leftrightarrow \mathcal{P} \qquad \mathcal{P} \vee (\mathcal{P} \wedge \mathcal{Q}) \Leftrightarrow \mathcal{P}$$

Simplification:

$$\mathcal{P} \wedge (\mathcal{Q} \vee \neg\mathcal{Q}) \Leftrightarrow \mathcal{P} \qquad \mathcal{P} \vee (\mathcal{Q} \wedge \neg\mathcal{Q}) \Leftrightarrow \mathcal{P}$$

$$\mathcal{P} \vee (\mathcal{Q} \vee \neg\mathcal{Q}) \Leftrightarrow (\mathcal{Q} \vee \neg\mathcal{Q}) \qquad \mathcal{P} \wedge (\mathcal{Q} \vee \neg\mathcal{Q}) \Leftrightarrow \mathcal{P}$$

Simplifying sentences

- $([B \vee (\neg A \vee \neg B)] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$

Simplifying sentences

- $([B \vee (\neg A \vee \neg B)] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$
- $([(\neg A \vee \neg B) \vee B] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Comm

Simplifying sentences

- $([B \vee (\neg A \vee \neg B)] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$
- $([(\neg A \vee \neg B) \vee B] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Comm
- $([\neg A \vee (\neg B \vee B)] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Assoc

Simplifying sentences

- $([B \vee (\neg A \vee \neg B)] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$
- $([(\neg A \vee \neg B) \vee B] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Comm
- $([\neg A \vee (\neg B \vee B)] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Assoc
- $([\neg B \vee B] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Simpl

Simplifying sentences

- $([B \vee (\neg A \vee \neg B)] \wedge [B \vee (A \vee B)]) \wedge$
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  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Comm
- $([\neg A \vee (\neg B \vee B)] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Assoc
- $([\neg B \vee B] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Simpl
- $[B \vee (A \vee B)] \wedge ([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Simpl

Simplifying sentences

- $([B \vee (\neg A \vee \neg B)] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$
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  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Comm
- $([\neg A \vee (\neg B \vee B)] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Assoc
- $([\neg B \vee B] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Simpl
- $[B \vee (A \vee B)] \wedge ([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Simpl
- $[(A \vee B) \vee B] \wedge ([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Comm

Simplifying sentences

- $([B \vee (\neg A \vee \neg B)] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$
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  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Comm
- $([\neg A \vee (\neg B \vee B)] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Assoc
- $([\neg B \vee B] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Simpl
- $[B \vee (A \vee B)] \wedge ([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Simpl
- $[(A \vee B) \vee B] \wedge ([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Comm
- $[A \vee (B \vee B)] \wedge ([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Assoc

Simplifying sentences

- $([B \vee (\neg A \vee \neg B)] \wedge [B \vee (A \vee B)]) \wedge$
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  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Comm
- $([\neg A \vee (\neg B \vee B)] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Assoc
- $([\neg B \vee B] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Simpl
- $[B \vee (A \vee B)] \wedge ([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Simpl
- $[(A \vee B) \vee B] \wedge ([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Comm
- $[A \vee (B \vee B)] \wedge ([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Assoc
- $[A \vee B] \wedge ([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Idem

Simplifying sentences

- $([B \vee (\neg A \vee \neg B)] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$
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  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Comm
- $([\neg A \vee (\neg B \vee B)] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Assoc
- $([\neg B \vee B] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Simpl
- $[B \vee (A \vee B)] \wedge ([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Simpl
- $[(A \vee B) \vee B] \wedge ([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Comm
- $[A \vee (B \vee B)] \wedge ([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Assoc
- $[A \vee B] \wedge ([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Idem
- $([A \vee B] \wedge [(A \vee B) \vee \neg C]) \wedge [(\neg A \vee \neg B) \vee \neg C]$ Comm,Assoc

Simplifying sentences

- $([B \vee (\neg A \vee \neg B)] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$
- $([(\neg A \vee \neg B) \vee B] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Comm
- $([\neg A \vee (\neg B \vee B)] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Assoc
- $([\neg B \vee B] \wedge [B \vee (A \vee B)]) \wedge$
  $([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Simpl
- $[B \vee (A \vee B)] \wedge ([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Simpl
- $[(A \vee B) \vee B] \wedge ([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Comm
- $[A \vee (B \vee B)] \wedge ([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Assoc
- $[A \vee B] \wedge ([(\neg A \vee \neg B) \vee \neg C] \wedge [(A \vee B) \vee \neg C])$ Idem
- $([A \vee B] \wedge [(A \vee B) \vee \neg C]) \wedge [(\neg A \vee \neg B) \vee \neg C]$ Comm,Assoc

Lecture 32

Wednesday, April 01, 2020

Short history of logic

History of logic

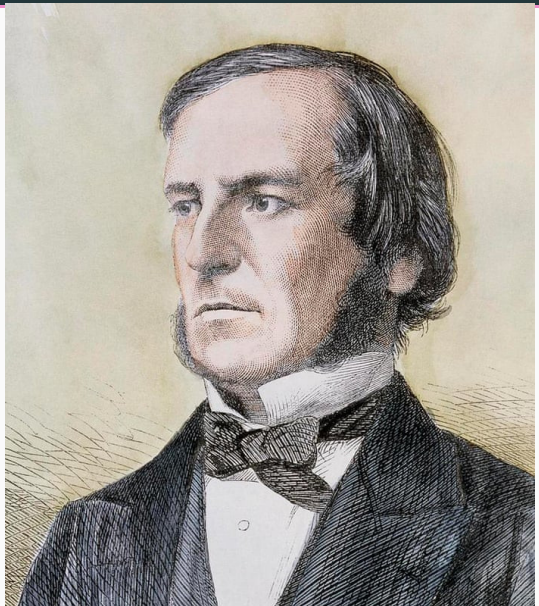
The beginnings: Aristotle

- Lived 384–322 BCE
- Cataloged valid arguments (“syllogisms”), e.g.,
- All ungulates have hooves.
  No fish have hooves.
  ∴ No fish are ungulates.

The middle ages

- Ibn Sīnā (Avicenna)
- Pierre Abelard
- William Ockham
- Jean Buridan

Mathematical logic: Boole et al.

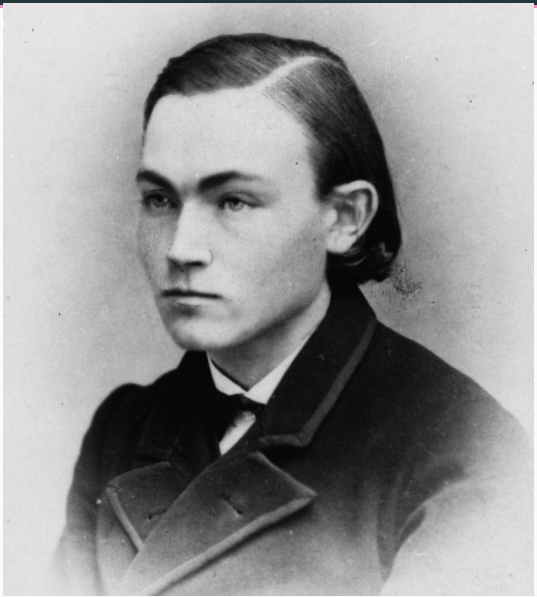


- George Boole
- John Venn
- Augustus De Morgan
- Charles Lutwidge Dodgson (aka Lewis Caroll)

The algebra of logic: Peirce at al

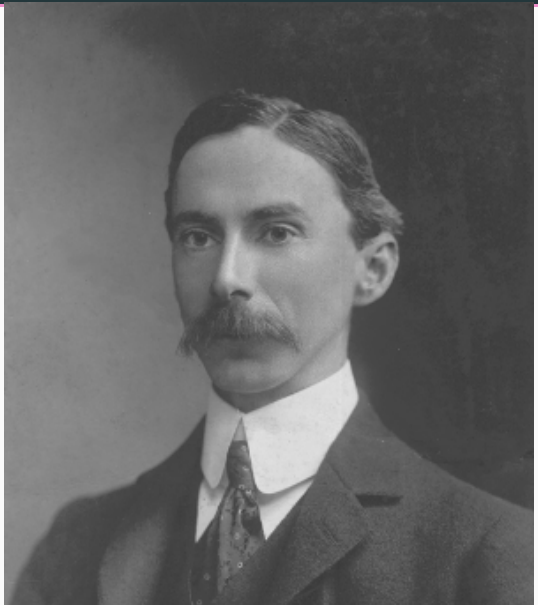
- Charles Sanders Peirce
- Christine Ladd–Franklin
- Ernst Schröder

Modern logic: Gottlob Frege



- 1848–1925
- Predicates and quantifiers
- Plan to turn all of math into theorems of logic alone

Modern logic: Bertrand Russell



- 1870–1972
- Showed Frege's system contradictory
- Fixed it (*Principia mathematica*, 3 vols.)
- Plan to turn all of math into theorems of logic alone

Modern logic: David Hilbert

- 1862–1943
- Combined Russell's and Schröder's systems
- First modern logic textbook
- Plan to turn all of math into consequences of a single set of premises

Modern logic: Kurt Gödel

- 1906–1978
- Showed that every valid argument has a proof
- Showed that Frege/Russell's and Hilbert's plans can't work

Modern logic: Alan Turing

- 1912–1954
- Showed that unlike TFL, FOL has no decision procedure
- Invented Turing machines (“father of computer science”)

Modern logic: Gerhard Gentzen

- 1909–1945
- Invented natural deduction
- Founded theory of proofs

Modern logic: modal logic

- Extend logic with operators for "possible" and "necessary"
- Pioneered by philosophers, now used by computer scientists
- Rudolf Carnap, Saul Kripke, Ruth Barcan Marcus

Lecture 33

Friday, April 03, 2020

Non-classical logics

Philosophy and nonstandard logics

Validity and validity in FOL

- Philosophers interested in **valid arguments**
- Definition: There is no case where the premises are true and the conclusion is false
  - **Important:** It does not say "it **isn't in fact the case** that the premises are true and the conclusion is false"
  - That would make every argument with
    - true premises, true conclusion
    - false premises, true conclusion
    - false premises, false conclusion

    valid. But that's not the case.
  - It says "it is **impossible** that the premises could be true and the conclusion false!"
- Difficulty: What logically possible circumstances are there?

What logic does for validity

- Truth–tables, interpretations, proofs give **sufficient conditions** for validity, i.e.,
  - Every argument valid in TFL is valid
  - Every argument valid in FOL is valid
  - Every argument with a formal proof is valid (soundness!)

Nonstandard logics

- Formal models of logical consequence make a number of simplifying assumptions:
  - Only **determinate** properties allowed, e.g, no vague properties
  - Every (atomic) sentence either **T** or **F**; not both and nothing in between
  - Every name must refer, i.e., no empty names
  - Only truth–functional connectives, e.g., no subjunctive contionals, “because”, or tenses
- Non–standard logics: expand TFL, FOL to deal with these

Many-valued logic

- Add to the truth-values **T** and **F**, e.g.,
  - "Undetermined": neither true nor false

P	$\neg P$
T	F
U	U
F	T

P	Q	$(P \wedge Q)$
T	T	T
T	U	U
T	F	F
U	T	U
U	U	U
U	F	F
F	T	F
F	U	F
F	F	F

P	Q	$(P \vee Q)$
T	T	T
T	U	T
T	F	T
U	T	T
U	U	U
U	F	U
F	T	T
F	U	U
F	F	F

  - "Inconsistent": both true and false
  - Fuzzy truth values: any number between 0 and 1

Truth-functional connectives

Definition

A connective $*$ is **truth functional** iff the truth value of $*A$ depends only on the truth value of A.

- "It is not the case that" is truth functional.
- So are "and", "or", "neither nor".
- "If . . . then": iffy.

Non-truth-functional connectives

- “Possibly”, “Necessarily”
- Subjunctive conditionals
- Tenses: “Is always true,” “Will be true,” “Was true”
- “Richard believes that”, “Richard knows that”

Possibly

- "It is possible that ...", "Possibly, ..."
- Consider:
  1. It is possible that I will live forever.
  2. It is possible that $2 + 2 = 5$.
- (1) is true and (2) is false.
- But $A_1 =$ "I will live forever" and $A_2 =$ "$2 + 2 = 5$" are both false.
- So "It is possible that A" can't just depend on the truth value of A
- Otherwise (1) and (2) would have to have the same truth value.

Subjunctive Conditionals

- Subjunctive conditionals = if—then statements in **subjunctive** mood
- "If P were true, then Q would be true."
- Indicative conditional is (plausibly) **truth-functional**: truth value of "If P, then Q" depends **only on truth values** of P and Q.

Subjunctive Conditionals

- Subjunctive conditional is not truth functional
- E.g., consider:
  1. If the world were just, no evil deed would go unpunished.
     P_1 = the world is just
     Q_1 = no evil deed goes unpunished
  2. If the world were flat, no evil deed would go unpunished.
     P_2 = the world is flat
     Q_2 = no evil deed goes unpunished
- P_1, Q_1 both false; P_2, Q_2 both false, but
- (1) is true, but (2) is false

Modal logic

- Alethic logic
  "It is possible that" ($\Diamond$), "it is necessary that" ($\Box$)

$$\Box A \to A \qquad \Diamond\Box A \to \Box A$$

- Epistemic logic: "Richard knows that" (K)

$$\mathbf{K}\, A \to A \qquad \mathbf{K}\, \neg\mathbf{K}\, A \to \neg\mathbf{K}\, A$$

- Conditional logic
  Subjunctive conditionals, "if it were true that . . . , then it would be true that — —" ($\Box\!\!\to$)

$$(A \mathrel{\Box\!\!\to} B) \to (A \to B)$$

- Temporal logic
  "It was true that" (P), "It will be true that" (F)

$$\mathbf{F}\,\mathbf{P}\, A \to (\mathbf{P}\, A \vee A \vee \mathbf{F}\, A)$$

Temporal logic

- "Always A": $\Box A$
- "Sometimes A": $\Diamond A$
- If always A, then A (now): $\Box A \to A$
- If A (now), then sometimes A: $A \to \Diamond A$
- Always A iff not sometimes not A
  $\Box A \leftrightarrow \neg\Diamond\neg A$.
- If always A and B, then always A or always B:
  $\Box(A \wedge B) \to (\Box A \wedge \Box B)$
- If always A or B, then always A or always B:
  $\Box(A \vee B) \to (\Box A \vee \Box B)$

Semantics for TFL

- Recall: valuations map sentence letters to truth values
- Given a valuation v, we can define if a sentence of TFL is true on v:
  - $v \models P$ iff $v(P) = \mathbf{T}$
  - $v \models \neg\mathscr{A}$ iff not $v \models \mathscr{A}$
  - $v \models (\mathscr{A} \vee \mathscr{B})$ iff $v \models \mathscr{A}$ or $v \models \mathscr{B}$.
  - $v \models (\mathscr{A} \wedge \mathscr{B})$ iff $v \models \mathscr{A}$ and $v \models \mathscr{B}$.
  - $v \models (\mathscr{A} \rightarrow \mathscr{B})$ iff either not $v \models \mathscr{A}$ or $v \models \mathscr{B}$.

Semantics for temporal logic

- Collection of points in time t
- For every time t, a valuation v_t
- Define $\mathscr{A}$ is true **at time t**:
  - $t \models P$ iff $v_t(P) = \mathbf{T}$
  - $t \models \neg\mathscr{A}$ iff not $t \models \mathscr{A}$
  - $t \models (\mathscr{A} \vee \mathscr{B})$ iff $t \models \mathscr{A}$ or $t \models \mathscr{B}$.
  - $t \models (\mathscr{A} \wedge \mathscr{B})$ iff $t \models \mathscr{A}$ and $t \models \mathscr{B}$.
  - $t \models (\mathscr{A} \rightarrow \mathscr{B})$ iff either not $t \models \mathscr{A}$ or $t \models \mathscr{B}$.
  - $t \models \Box\mathscr{A}$ iff, at all times s, $s \models \mathscr{A}$
  - $t \models \Diamond\mathscr{A}$ iff, at some time s, $s \models \mathscr{A}$

Validity

- Some sentences are always true, however we interpret sentence letters, e.g.,

$$\Box A \to \neg\Diamond\neg A$$

- Suppose at time t, $t \models \Box A$.
- Then at all times s, $s \models A$.
- So at no time s, $s \models \neg A$.
- So not: at some time s, $s \models \neg A$.
- So not: $t \models \Diamond\neg A$.
- Therefore: $t \models \neg\Diamond\neg A$.

Invalidity

- Some sentences are not always true
- Obviously, any sentence not involving $\Box$, $\Diamond$ that isn't a tautology
- But also, e.g., $\Box(A \vee B) \to (\Box A \vee \Box B)$
- Counterexample:
  - Two times, 1 and 2.
  - $v_1(A) = v_2(B) = \mathbf{T}$, $v_2(A) = v_2(B) = \mathbf{F}$
  - $1 \models A \vee B$ (since $1 \models A$)
  - $2 \models A \vee B$ (since $2 \models B$)
  - So at every time t, $t \models A \vee B$
  - So $1 \models \Box(A \vee B)$
  - But neither $1 \models \Box A$ nor $1 \models \Box B$

Lecture 34

Monday, April 06, 2020

Applications of logic

Semantics and proof theory

Semantics

- A **truth-value assignment** is an assignment of **T** or **F** to the sentence letters (schematic letters in the truth-functional form)
- An **interpretation** is a non-empty domain together with
  - extensions for each predicate symbol
  - objects in the domain for each constant symbol
  - functions for each function symbol
- A tautology is a sentence (the truth-functional form of) which is true in all truth-value assignments
- A validity is a sentence that's true in all interpretations

Soundness and Completeness

- Soundness
  Arguments have formal proofs only if they are valid
  If there is a proof of B from premises $A_1, \ldots A_n$, then B is a consequence of $A_1, \ldots A_n$
- Completeness
  Arguments have formal proofs if they are valid
  If $A_1, \ldots A_n$ entail B in FOL, then there is a proof of B from premises $A_1, \ldots A_n$

  Proved by Kurt Gödel (1929)

Other proof systems: resolution

- Natural deduction is a proof system for **validity/tautologies**
- Also possible to design proof systems for dual notion: **unsatisfiability**
  (A is a tautology $\Leftrightarrow \neg A$ is unsatisfiable)
- A **clause** is a disjunction $A_1 \vee \ldots \vee A_n$ where each A_i is atomic or negated atomic.
- CNF theorem: every sentence is equivalent to a conjunction of clauses.

Other proof systems: resolution

- The resolution rule:

$$\frac{A_1 \vee \ldots \vee A_n \vee C \qquad B_1 \vee \ldots \vee B_m \vee \neg C}{A_1 \vee \ldots \vee A_n \vee B_1 \vee \ldots \vee B_m}$$

- Preserves **joint satisfiability**
- If you can prove the empty clause $\bot$ from a set of clauses, they can't be jointly satisfiable.

Theories and decidability

Church–Turing Theorem

Instance: Sentence A of FOL
Problem: Is A a validity?

- Undecidable: no computer program can answer this question correctly for all A.
- Proved independently by Alonzo Church and Alan Turing in 1935

Cook's Theorem

Instance: Sentence A of TFL
Problem: Is A a tautology?

- Decidable: write a computer program that checks all valuation for A.
- But: it's hard: "co–NP complete"
- Proved independently by Stephen Cook (1971) and Leonid Levin (1973)

Decidable Classes

- The decision problem **in general** is undecidable
- But special cases **can** be decided, e.g.:

Instance: Sentence A with only 1–place predicate symbols
Problem: Is A a validity?

- Decidable
- Proved by Leopold Löwenheim (1915)
- Complexity is NEXPTIME–complete.

Theories

- A set of sentences of FOL also called a **theory**, and the sentences in it **axioms**
- Some (types of) interpretations can be characterized as those interpretations in which every sentence in the theory is true
- Examples:
  - Mathematical theories (theory of orders, group theory, arithmetic)
  - KR classification systems, e.g., SNOMED-CT
  - Mereology, theories of truth, scientific theories

The axiomatic method

- Theories + logic: what follows from axioms?
- Axiomatic method: do science by investigating what follows from the axioms of a theory
- Logic can also determine:
  - Are axioms (in)consistent?
  - Are axioms independent, or is one superfluous?
- Paradigm of axiomatic method: geometry (Euclid)

Examples of theories: linear orders

A relation $\preceq$ on a set O is a **linear order** iff it makes following axioms true:

$$\forall x \forall y ((x \preceq y \wedge y \preceq x) \rightarrow x = y) \qquad \text{Antisymmetry}$$

$$\forall x \forall y \forall z ((x \preceq y \wedge y \preceq z) \rightarrow x \preceq z) \qquad \text{Transitivity}$$

$$\forall x \forall y (x \preceq y \vee y \preceq x) \qquad \text{Totality}$$

Every total relation is reflexive:

$$LO \models \forall x \, x \preceq x$$

Examples of theories: Robinson's Q

Theories of arithmetic, such as Robinson's theory Q:

$$\neg\exists x\,(x+1) = 0$$

$$\forall x(x = 0 \vee \exists y\,(y+1) = x)$$

$$\forall x \forall y((x+1) = (y+1) \rightarrow x = y)$$

$$\forall x\,(x+0) = x$$

$$\forall x \forall y\,(x+(y+1)) = ((x+y)+1)$$

$$\forall x\,(x \times 0) = 0$$

$$\forall x \forall y\,(x \times (y+1)) = ((x \times y)+x)$$

Examples of theories: SNOMED-CT

```
bacterial pneumonia =
          is-a|bacterial infectious disease
          is-a|infective pneumonia
          causative agent|bacteria
          finding site|lung structure
```

$$\forall x(BacterialPneumonia(x) \leftrightarrow$$
$$BacterialInfectiousDisease(x) \wedge$$
$$InfectivePneumonia(x) \wedge$$
$$\exists y(HasCausativeAgent(x, y) \wedge Bacteria(y)) \wedge$$
$$\exists y(HasFindingSite(x, y) \wedge LungStructure(y)))$$

Examples of theories: SNOMED-CT

- Over 300,000 concepts (predicate symbols), e.g.,
  - 1-place predicates:
    parts of body, findings, organisms, physical objects, procedures, substances, diseases, ...
  - 2-place predicates:
    has finding site, has causative agent, with method, has active ingredient, laterality is, using device, ...
- About 1,000,000 descriptions (axioms)
- SNOMED-CT is decidable

Examples of Theories: Mereology

- Mereology: the theory of the part–whole relation (**metaphysics**)
- Primitive relation: $Pt(x, y)$, "x is a part of y"
- Some axioms:

$\forall x\, Pt(x, x)$	Reflexivity
$\forall x \forall y \forall z ((Pt(x, y) \wedge Pt(y, z)) \rightarrow Pt(x, z))$	Transitivity
$\forall x \forall y ((Pt(x, y) \wedge Pt(y, x)) \rightarrow x = y)$	Antisymmetry

Examples of Theories: Mereology

- Defined properties and relations

$$PP(x, y) \leftrightarrow Pt(x, y) \wedge \neg x = y$$
$$At(x) \leftrightarrow \neg\exists y\, PP(y, x)$$

- Different theories settle questions differently, e.g.,
  - Are there atoms?
  - Does everything comprise at least one atom?
  - Is everything made of atomless “gunk”?

Property theories and Grelling's paradox

- Primitive relation: $Ap(x, y)$, "x applies to y"
- Proposed axiom ("comprehension"): For any formula $P(y)$,

$$\exists x \forall y (Ap(x, y) \leftrightarrow P(y))$$

- Axiom is **inconsistent** (contradictory)
- A property is **heterological** if it does not apply to itself, i.e., $\neg Ap(x, x)$
- Is the property of being heterological itself heterological?

$$\begin{array}{|l} \forall y (Ap(h, y) \leftrightarrow \neg Ap(y, y)) \\ \hline Ap(h, h) \wedge \neg Ap(h, h) \end{array}$$

Completeness of theories

- A theory T is **complete** if for every sentence A in its language, either $T \models A$ or $T \models \neg A$
- Every complete theory is decidable!
- Some incomplete theories are still decidable (e.g., LO)
- Some incomplete theories are incomplete**able**: no consistent extension is complete
- Philosophical upshot of this: truth in the intended interpretation(s) of the theory outstrips provability from the theory

Gödel's Incompleteness Theorem (1930)
Arithmetic, set theory, mereology are incompleteable

Christmas party trick

If the first sentence on this slide is true,
then Santa Claus exists.

1. Assumption: The first sentence on this slide is true.

Christmas party trick

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1. Assumption: The first sentence on this slide is true.
2. If the first sentence on this slide is true, then Santa Claus exists (from (1), since if S is true, then S).

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3. Santa Claus exists
   (from (1) and (2), by modus ponens)

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4. If the first sentence on this slide is true, Santa Claus exists
   (from (1)–(3), by conditional intro).

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5. The first sentence on this slide is true.
   ((4) = the first sentence on this slide)

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3. Santa Claus exists
   (from (1) and (2), by modus ponens)
4. If the first sentence on this slide is true, Santa Claus exists
   (from (1)–(3), by conditional intro).
5. The first sentence on this slide is true.
   ((4) = the first sentence on this slide)
6. Santa Claus exists.
   (from (4) and (5), by modus ponens)

Lecture 35
Monday, April 13, 2020
Test 3

This is a test!

- If you think you'll leave early, sit near the aisle or in the front row.
- Do not disturb your colleagues when you leave.
- **Do not leave during the last 10 minutes.**
- Set your phone to silent and put it away.
- Don't open the test until I tell you to.
- Put your name and ID on the front page.
- Read the instructions.
- The only things out should be the test, writing implement, and hydration/snacks.
- No cheating!
- You have 50 minutes to complete the test.
- Good luck!