Hypersequents and Cut Elimination for Intuitionistic Fuzzy Logic*

Matthias Baaz[1] and Richard Zach[2]

[1] Institut für Algebra und Computermathematik E118.2,
Technische Universität Wien, A–1040 Vienna, Austria, baaz@logic.at
[2] Institut für Computersprachen E185.2,
Technische Universität Wien, A–1040 Vienna, Austria, zach@logic.at

Abstract. Takeuti and Titani have introduced and investigated a logic they called intuitionistic fuzzy logic. This logic is characterized as the first-order Gödel logic based on the truth value set $[0, 1]$. The logic is known to be axiomatizable, but no deduction system amenable to proof-theoretic, and hence, computational treatment, has been known. Such a system is presented here, based on previous work on hypersequent calculi for propositional Gödel logics by Avron. It is shown that the system is sound and complete, and allows cut-elimination. A question by Takano regarding the eliminability of the Takeuti-Titani density rule is answered affirmatively.

1 Introduction

Intuitionistic fuzzy logic **IF** was originally defined by Takeuti and Titani to be the logic of the complete Heyting algebra $[0, 1]$. In standard many-valued terminology, **IF** is $[0, 1]$-valued first-order Gödel logic, with truth functions as defined below. The finite-valued propositional versions of this logic were introduced by Gödel [8], and have spawned a sizeable area of logical research subsumed under the title "intermediate logics" (intermediate between classical and intuitionistic logic). The infinite-valued propositional Gödel logic was studied by Dummett [6], who showed that it is axiomatized by **LC**, i.e., intuitionistic propositional logic plus the linearity axiom $(A \supset B) \vee (B \supset A)$.

Takeuti and Titani [13] characterized **IF** by a calculus which extends the intuitionistic predicate calculus **LJ** by several axioms as well as the density rule

$$\frac{\Gamma \vdash A \vee (C \supset p) \vee (p \supset B)}{\Gamma \vdash A \vee (C \supset B)}\ tt'$$

This rule can be read as expressing the fact that the set of truth values is densely ordered. In this sense, the Takeuti-Titani axiomatization is the natural axiomatization of the $[0, 1]$-valued Gödel logic. The valid formulas of **IF** are

2000 Mathematics Subject Classification: Primary 03B50; Secondary 03B55, 03F05.

* Research supported by the Austrian Science Fund under grant P–12652 MAT

Clote, Peter G., and Helmut Schwichtenberg (eds.), *Computer Science Logic. 14th International Workshop, CSL 2000.* Fischbachau, Germany, August 21–26, 2000. Proceedings. LNCS 1862. Springer, Berlin, 2000. pp. 187–201.

also characterized as those formulas valid in *every* first-order Gödel logic based on a linearly ordered set of truth-values (this is obvious for all logics based on truth value sets $\subseteq [0,1]$, since a countermodel in such a truth-value set can be straightforwardly embedded in $[0,1]$. The general claim was established by Horn [10]). In this characterization, the density rule is not a natural assumption, since not every linearly ordered truth-value set is densely ordered. It follows from this characterization that the density rule is redundant for the axiomatization of **IF**, and completeness proofs without it have been given by Horn [10] and Takano [11].[1] Takano posed the question of whether a syntactic elimination of the density rule is also possible.

More recently, another axiomatizable first-order extension of **LC** has been studied by Corsi [4, 5] and Avellone et al. [1]. This extension is defined not via many-valued semantics but as the class of formulas valid in all linearly ordered intuitionistic Kripke models. It is different from **IF**; specifically, the formula ($\vee\forall$) below is not valid in it. **IF** can, however, also be characterized as the set of formulas valid in all linearly ordered Kripke models with constant domains (this was first observed by Gabbay [7, §3]).

The interest of **IF** lies in the fact that it combines properties of logics for approximate reasoning with properties of intuitionistic logic. On the one hand, **IF** is one of the basic t-norm logics (see Hájek [9]), on the other, it is an extension of intuitionistic logic which corresponds to concurrency (as has been argued by Avron [2]). We present here a calculus for **IF** which is adequate for further proof-theoretic study. The basic result in this regard is the cut-elimination theorem for this calculus, from which a midhypersequent-theorem can be derived. This theorem, in turn, corresponds to Herbrand's Theorem in classical logic, and as such is a possible basis for automated theorem proving in **IF**.

The calculus also allows us to investigate the proof-theoretic effects of the Takeuti-Titani rule. We give a positive answer to Takano's question, showing that the density rule can be eliminated from **IF**-proofs. A simple example illustrates the possible structural differences between proofs with and without the Takeuti-Titani rule.

2 Syntax and Semantics of Intuitionistic Fuzzy Logic

The language L of **IF** is a usual first-order language with propositional variables and where free (a, b, ...) and bound (x, y, ...) variables are distinguished.

Definition 1. An **IF**-*interpretation* $\Im = \langle D, \mathbf{s}\rangle$ is given by the *domain* D and the *valuation function* $\mathbf{s}$. Let L^D be L extended by constants for each element of D. Then $\mathbf{s}$ maps atomic formulas in $\mathrm{Frm}(L^D)$ into $[0,1]$, $d \in D$ to itself, n-ary function symbols to functions from D^n to D, and free variables to elements of D.

The valuation function $\mathbf{s}$ can be extended in the obvious way to a function on all terms. The valuation for formulas is defined as follows:

[1] Note that the corresponding axiom $(\forall p)((A \supset p) \vee (p \supset B)) \supset (A \supset B)$ is not redundant in quantified *propositional* $[0,1]$-valued Gödel logic. See [3].

1. $A \equiv P(t_1, \ldots, t_n)$ is atomic: $\Im(A) = \mathbf{s}(P)(\mathbf{s}(t_1), \ldots, \mathbf{s}(t_n))$.
2. $A \equiv \neg B$:
$$\Im(\neg B) = \begin{cases} 0 & \text{if } \Im(B) \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$
3. $A \equiv B \wedge C$: $\Im(B \wedge C) = \min(\Im(B), \Im(C))$.
4. $A \equiv B \vee C$: $\Im(B \vee C) = \max(\Im(A), \Im(B))$.
5. $A \equiv B \supset C$:
$$\Im(B \supset C) = \begin{cases} \Im(C) & \text{if } \Im(B) > \Im(C) \\ 1 & \text{if } \Im(B) \leq \Im(C). \end{cases}$$

The set $\text{Distr}_{\Im}(A(x)) = \{\Im(A(d)) : d \in D\}$ is called the *distribution* of $A(x)$. The quantifiers are, as usual, defined by infimum and supremum of their distributions.

(6) $A \equiv (\forall x)B(x)$: $\Im(A) = \inf \text{Distr}_{\Im}(B(x))$.
(7) $A \equiv (\exists x)B(x)$: $\Im(A) = \sup \text{Distr}_{\Im}(B(x))$.

$\Im$ *satisfies* a formula A, $\Im \models A$, if $\Im(A) = 1$. A formula A is **IF**-valid if every **IF**-interpretation satisfies it.

Note that, as in intuitionistic logic, $\neg A$ may be defined as $A \supset \bot$, where $\bot$ is some formula that always takes the value 0.

3 Hypersequents and IF

Takeuti and Titani's system **IF** is based on Gentzen's sequent calculus **LJ** for intuitionistic logic with a number of extra axioms

$$\vdash (A \supset B) \vee ((A \supset B) \supset B) \tag{Ax1}$$
$$(A \supset B) \supset B \vdash (B \supset A) \vee B \tag{Ax2}$$
$$(A \wedge B) \supset C \vdash (A \supset C) \vee (B \supset C) \tag{Ax3}$$
$$(A \supset (B \vee C)) \vdash (A \supset B) \vee (A \supset C) \tag{Ax4}$$
$$(\forall x)(A(x) \vee B) \vdash (\forall x)A(x) \vee B \tag{$\vee\forall$}$$
$$(\forall x)A(x) \supset C \vdash (\exists x)(A(x) \supset D) \vee (D \supset C) \tag{$\forall\supset$}$$

(where x does not occur in B or D) and the following additional inference rule:

$$\frac{\Gamma \vdash A \vee (C \supset p) \vee (p \supset B)}{\Gamma \vdash A \vee (C \supset B)}\ tt'$$

where p is a propositional eigenvariable (i.e., it does not occur in the lower sequent). It is known that the extra inference rule is redundant. In fact, the system **H** of Horn [10] consisting of **LJ** plus the schemata

$$(\forall x)(A(x) \vee B) \supset (\forall x)A(x) \vee B \tag{$\vee\forall$}$$
$$(A \supset B) \vee (B \supset A) \tag{D}$$

is complete for **IF** (see also [11]). Neither of these systems, however, has decent proof-theoretic properties such as cut elimination, nor is a syntactic method for the elimination of the Takeuti-Titani rule (tt') known. Takano [11] has posed the question of a syntactic elimination procedure of the Takeuti-Titani rule as an open problem.

We present a system which has the required properties, and which allows the syntactic elimination of the Takeuti-Titani rule. Our system is based on Avron's [2] cut-free axiomatization of **LC** using a hypersequent calculus.

Definition 2. A *sequent* is an expression of the form

$$\Gamma \vdash \Delta$$

where Γ and Δ are finite multisets of formulas, and Δ contains at most one formula. A *hypersequent* is a finite multiset of sequents, written as

$$\Gamma_1 \vdash \Delta_1 \mid \ldots \mid \Gamma_n \vdash \Delta_n$$

The hypersequent calculus **HIF** has the following axioms and rules:
Axioms: $A \vdash A$, for any formula A.
Internal structural rules:

$$\frac{G \mid \Gamma \vdash \Delta}{G \mid A, \Gamma \vdash \Delta}\ iw\vdash \qquad \frac{G \mid \Gamma \vdash}{G \mid \Gamma \vdash A}\ \vdash iw \qquad \frac{G \mid A, A, \Gamma \vdash \Delta}{G \mid A, \Gamma \vdash \Delta}\ ic\vdash$$

External structural rules:

$$\frac{G}{G \mid \Gamma \vdash \Delta}\ ew \qquad \frac{G \mid \Gamma \vdash \Delta \mid \Gamma \vdash \Delta}{G \mid \Gamma \vdash \Delta}\ ec$$

Logical rules:

$$\frac{G \mid \Gamma \vdash A}{G \mid \neg A, \Gamma \vdash}\ \neg\vdash \qquad \frac{G \mid A, \Gamma \vdash}{G \mid \Gamma \vdash \neg A}\ \vdash\neg$$

$$\frac{G \mid A, \Gamma \vdash \Delta \quad G \mid B, \Gamma \vdash \Delta}{G \mid A \vee B, \Gamma \vdash \Delta}\ \vee\vdash \qquad \frac{G \mid \Gamma \vdash A \quad G \mid \Gamma \vdash B}{G \mid \Gamma \vdash A \wedge B}\ \vdash\wedge$$

$$\frac{G \mid \Gamma \vdash A}{G \mid \Gamma \vdash A \vee B}\ \vdash\vee_1 \qquad \frac{G \mid A, \Gamma \vdash \Delta}{G \mid A \wedge B, \Gamma \vdash \Delta}\ \wedge\vdash_1$$

$$\frac{G \mid \Gamma \vdash B}{G \mid \Gamma \vdash A \vee B}\ \vdash\vee_2 \qquad \frac{G \mid B, \Gamma \vdash \Delta}{G \mid A \wedge B, \Gamma \vdash \Delta}\ \wedge\vdash_2$$

$$\frac{G \mid \Gamma_1 \vdash A \quad G \mid B, \Gamma_2 \vdash \Delta}{G \mid A \supset B, \Gamma_1, \Gamma_2 \vdash \Delta}\ \supset\vdash \qquad \frac{G \mid A, \Gamma \vdash B}{G \mid \Gamma \vdash A \supset B}\ \vdash\supset$$

$$\frac{G \mid A(t), \Gamma \vdash \Delta}{G \mid (\forall x)A(x), \Gamma \vdash \Delta}\ \forall\vdash \qquad \frac{G \mid \Gamma \vdash A(a)}{G \mid \Gamma \vdash (\forall x)A(x)}\ \vdash\forall$$

$$\frac{G \mid A(a), \Gamma \vdash \Delta}{G \mid (\exists x)A(x), \Gamma \vdash \Delta}\ \exists\vdash \qquad \frac{G \mid \Gamma \vdash A(t)}{G \mid \Gamma \vdash (\exists x)A(x)}\ \vdash\exists$$

Cut:

$$\frac{G \mid \Gamma \vdash A \quad G \mid A, \Pi \vdash \Lambda}{G \mid \Gamma, \Pi \vdash \Lambda}\ cut$$

Communication:

$$\frac{G \mid \Theta_1, \Theta_1' \vdash \Xi_1 \quad G \mid \Theta_2, \Theta_2' \vdash \Xi_2}{G \mid \Theta_1, \Theta_2' \vdash \Xi_1 \mid \Theta_1', \Theta_2 \vdash \Xi_2}\ cm$$

Density:

$$\frac{G \mid \Phi \vdash p \mid p, \Psi \vdash \Sigma}{G \mid \Phi, \Psi \vdash \Sigma}\ tt$$

The rules $(\vdash \forall)$, $(\exists \vdash)$, and (tt) are subject to eigenvariable conditions: the free variable a and the propositional variable p, respectively, must not occur in the lower hypersequent. We denote the calculus obtained from **HIF** by omitting the cut rule by $\mathbf{HIF}^-$, and that obtained by omitting (tt) by $\mathbf{HIF}^*$.

The semantics of **IF** can easily be extended to hypersequents by mapping a hypersequent H

$$\Gamma_1 \vdash \Delta_1 \mid \ldots \mid \Gamma_n \vdash \Delta_n$$

to the formula H^*

$$(\bigwedge \Gamma_1 \supset \bigvee \Delta_1) \vee \ldots \vee (\bigwedge \Gamma_n \supset \bigvee \Delta_n)$$

where $\bigwedge \Gamma_i$ denotes the conjunction of the formulas in Γ_i or $\top$ if Γ_i is empty, and $\bigvee \Delta_i$ the disjunction of the formulas in Δ_i or $\bot$ if Δ_i is empty. Deriving a formula A in **HIF** then is equivalent to deriving the sequent $\vdash A$: the translation of $\vdash A$, i.e., $\top \supset A$ is equivalent to A.

Theorem 3 (Soundness). *Every hypersequent H derivable in* **HIF** *is* **IF**-*valid.*

Proof. By induction on the length of the proof. It will suffice to show that the axioms are valid, and that the quantifier rules and (tt) preserve validity.

The soundness of the quantifier rules is established by observing that corresponding quantifier shifting rules are intuitionistically valid. For instance, since

$$(\exists x)(B \vee A(x)) \supset (B \vee (\exists x)A(x)) \qquad (\vee\exists)$$

$$(\exists x)(B \supset A(x)) \supset B \supset (\exists x)A(x) \qquad (\supset\exists)$$

are intuitionistically valid, it is easily seen that $\vdash\exists$ is a sound rule. The only problematic rules are $(\vdash\forall)$ and $(\exists\vdash)$. Suppose $G \mid \Gamma \vdash A(a)$ is derivable in **HIF**. By induction hypothesis, $G^* \vee (\bigwedge \Gamma \supset A(a))$ is valid. Then certainly $(\forall x)(G^* \vee (\bigwedge \Gamma \supset A(x)))$ is **IF**-valid. Since a did not occur in G or Γ, we may now assume that x does not either. Since the quantifier shift $(\vee\forall)$, i.e.,

$$(\forall x)(B \vee A(x)) \supset (B \vee (\forall x)A(x)),$$

is valid in **IF**, we see that $G^* \vee (\forall x)(\bigwedge \Gamma \supset A(x))$ is valid. The result follows since

$$(\forall x)(B \supset A(x)) \supset B \supset (\forall x)A(x)$$

is intuitionistically valid, and hence **IF**-valid.

The communication rule is sound as well. Suppose the interpretation $\Im$ satisfies the premises of (cm). The only case where the conclusion is not obviously also satisfied is if $\Im(\Theta_1') \leq \Im(\Sigma_1)$ and $\Im(\Theta_2') \leq \Im(\Sigma_2)$. If the left lower sequent is not satisfied, we have $\Im(\Sigma_1) < \Im(\Theta_2')$, and hence $\Im(\Theta_1') \leq \Im(\Sigma_2)$, and thus the right lower sequent is satisfied. Similarly if the right lower sequent is not satisfied.

For (tt) we may argue as follows: Suppose that the hypersequent

$$H = G \mid \Phi \vdash p \mid p, \Psi \vdash \Sigma$$

is **IF**-valid. Let $\Im$ be an interpretation, and let $\Im_r$ be just like $\Im$ except that $\Im(p) = r$. Since p does not occur in the conclusion hypersequent

$$H' = G \mid \Phi, \Psi \vdash \Sigma$$

we have $\Im(H') = \Im_r(H')$ and $\Im(G) = \Im_r(G)$. If $\Im \models G$ we are done. Otherwise, assume that $\Im \not\models H'$, i.e.,

$$r_1 = \min\{\Im(\Phi), \Im(\Psi)\} > \Im(\Sigma) = r_2$$

Let $r = (r_1 + r_2)/2$. Now consider $\Im_r$: $\Im_r \not\models G$ by assumption; $\Im_r \not\models \Phi \vdash p$, since $\Im_r(\Phi) > r$; and $\Im_r \not\models p, \Psi \vdash \Sigma$, since $\Im_r(\Psi) > r > \Im_r(\Sigma)$. Hence, $\Im_r \not\models H$, a contradiction. □

Theorem 4 (Completeness). *Every* **IF***-valid hypersequent is derivable in* **HIF***.*

Proof. Observe that a hypersequent H and its canonical translation $\vdash H^*$ are interderivable using the cut rule and the following derivable hypersequents

$$A \vee B \vdash A \mid A \vee B \vdash B \qquad A \supset B, A \vdash B$$

$$A \wedge B \vdash A \qquad\qquad A \vdash A \vee B$$

Thus it suffices to show that the characteristic axioms of **IF** are derivable; a simple induction on the length of proofs shows that proofs in intuitionistic predicate calculus together with the axioms (D) and $(\vee\forall)$ can be simulated in **HIF**. The formula (D) is easily derivable using the communication rule.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A \vdash B \mid B \vdash A}\;cm}{\vdash A \supset B \mid B \vdash A}\;\vdash\supset}{\vdash A \supset B \mid\ \vdash B \supset A}\;\vdash\supset}{\vdash (A \supset B) \vee (B \supset A) \mid\ \vdash B \supset A}\;\vdash\vee}{\vdash (A \supset B) \vee (B \supset A) \mid\ \vdash (A \supset B) \vee (B \supset A)}\;\vdash\vee}{\vdash (A \supset B) \vee (B \supset A)}\;ec$$

The formula $(\vee\forall)$ can be obtained thus:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a)\vdash A(a)\quad B\vdash B}{B\vdash A(a)\mid A(a)\vdash B}\,cm\qquad \dfrac{B\vdash B}{B\vdash A(a)\mid B\vdash B}\,ew}{B\vdash A(a)\mid B\vee A(a)\vdash B}\,\vee\vdash\qquad \dfrac{A(a)\vdash A(a)}{A(a)\vdash A(a)\mid B\vee A(a)\vdash B}\,ew}{B\vee A(a)\vdash A(a)\mid B\vee A(a)\vdash B}\,\vee\vdash}{(\forall x)(B\vee A(x))\vdash A(a)\mid B\vee A(a)\vdash B}\,\forall\vdash}{(\forall x)(B\vee A(x))\vdash A(a)\mid (\forall x)(B\vee A(x))\vdash B}\,\forall\vdash}{(\forall x)(B\vee A(x))\vdash (\forall x)A(x)\mid (\forall x)(B\vee A(x))\vdash B}\,\vdash\forall}{(\forall x)(B\vee A(x))\vdash B\vee(\forall x)A(x)}\,\vdash\vee$$

The last line is obtained from the preceding by two $(\vdash\vee)$ inferences, followed by an external contraction. We indicate this with the double inference line. ☐

Of course, the other axioms of Takeuti's and Titani's system are also derivable. We will leave the propositional axioms 1–4 as an exercise to the reader, and give the derivation on of $(\forall\supset)$ as another example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a)\vdash A(a)\quad D\vdash D}{A(a)\vdash D\mid D\vdash A(a)}\,cm}{\vdash A(a)\supset D\mid D\vdash A(a)}\,\vdash\supset}{\vdash(\exists x)(A(x)\supset D)\mid D\vdash A(a)}\,\vdash\exists}{\vdash(\exists x)(A(x)\supset D)\mid D\vdash(\forall x)A(x)}\,\vdash\forall\qquad \dfrac{C\vdash C}{\vdash(\exists x)(A(x)\supset D)\mid C\vdash C}\,ew}{\vdash(\exists x)(A(x)\supset D)\mid(\forall x)A(x)\supset C, D\vdash C}\,\supset\vdash}{\vdash(\exists x)(A(x)\supset D)\mid(\forall x)A(x)\supset C\vdash D\supset C}\,\vdash\supset}{(\forall x)A(x)\supset C\vdash(\exists x)(A(x)\supset D)\vee(D\supset C)}\,\vdash\vee$$

4 Cut Elimination and Midhypersequent Theorem

Theorem 5 (Cut Elimination). *Any derivation of a hypersequent G in* **HIF** *can be transformed into a derivation of G in* $\mathbf{HIF}^-$.

This theorem is proved in the usual way by induction on the number of applications of the cut rule, using the following lemma.

Lemma 6. *Suppose the hypersequents*

$$H_1 = G\mid\Gamma\vdash A\quad\textit{and}\quad H_2 = G\mid\Pi\vdash\Lambda$$

are cut-free derivable. Then

$$H = G\mid\Gamma,\Pi^*\vdash\Lambda$$

where Π^* is obtained from Π by removing all occurrences of A, is cut-free provable, and the number of applications of (ec) in the resulting proof is not more than the sum of applications of (ec) in γ and δ.

Proof. Let γ and δ be the cut-free proofs of G and H, respectively. We may assume, renaming variables if necessary, that the eigenvariables in γ and δ are distinct. The proof follows Gentzen's original *Hauptsatz*. Define the following measures on the pair $\langle\gamma, \delta\rangle$: the *rank* $r = \mathrm{len}(\gamma) + \mathrm{len}(\delta)$, the *degree* $d = \deg(A)$, and the *order* o is the number of applications of the (ec) rule in γ, δ. We proceed by induction on the lexicographical order of $\langle d, o, r\rangle$.

If either H_1 or H_2 is an axiom, then H can be derived from H_1 or H_2, respectively, using only weakenings. (This includes the case where $r = 2$).

Otherwise, we distinguish cases according to the last inferences in γ and δ. The induction hypothesis is that the claim of the lemma is true whenever the degree is $< d$ or is $= d$ and either the order $< o$, or the order $= o$ and the rank $< r$.

(1) γ or δ ends in an inference which acts on a sequent in G. We may invoke the induction hypothesis on the premises of H_1 or H_2, and H_2 or G_2, respectively.

(2) γ or δ ends in (ec). For instance, γ ends in

$$\dfrac{\begin{array}{c}\vdots\ \gamma' \\ G \mid \Gamma \vdash A \mid \Gamma \vdash A\end{array}}{G \mid \Gamma \vdash A}\ ec$$

Apply the induction hypothesis to γ' and δ. The resulting proof γ'' of

$$G \mid \Gamma \vdash A \mid \Gamma, \Pi^* \vdash \Lambda$$

has one less (ec) than γ (although it may be much longer), and so the induction hypothesis applies again to γ'' and δ.

(3) γ or δ end in another structural inference, (tt), or (cm): These cases are unproblematic applications of the induction hypothesis to the premises, followed by applications of structural inferences.

For example, assume γ ends in (cm), i.e.,

$$\dfrac{\begin{array}{c}\vdots\ \gamma_1 \\ G \mid \Theta_1, \Theta_1' \vdash \Xi_1\end{array} \quad \begin{array}{c}\vdots\ \gamma_2 \\ G \mid \Theta_2, \Theta_2' \vdash A\end{array}}{G \mid \Theta_1, \Theta_2' \vdash \Xi_1 \mid \Theta_1', \Theta_2 \vdash A}\ cm$$

where $\Gamma = \Theta_1', \Theta_2$. Apply the deduction hypothesis to the right premise and H_2 to obtain a cut-free proof of

$$G \mid \Theta_2, \Theta_2', \Pi^* \vdash \Lambda$$

Using applications of (ew) and (cm), we obtain the desired result.

The case of (tt) may be of special interest. Suppose γ ends in(tt), with

$$\dfrac{G \mid \Phi \vdash p \mid p, \Psi \vdash A}{G \mid \Phi, \Psi \vdash A}\ tt$$

Apply the induction hypothesis to the premises of H_1 and H_2, and apply (tt) to obtain the desired proof:

$$\frac{G \mid \Phi \vdash p \mid p, \Psi, \Pi^* \vdash \Lambda}{G \mid \Phi, \Psi, \Pi^* \vdash \Lambda}\ tt$$

The case of δ ending in (tt) is handled similarly.

(4) γ ends in a logical inference not involving the cut formula, or δ ends in a logical inference not involving the cut formula. These cases are easily handled by appeal to the induction hypothesis and application of appropriate logical and structural inferences. We outline the case where γ ends in $(\supset\vdash)$:

$$\frac{\begin{array}{c}\vdots\ \gamma_1 \\ G \mid C, \Gamma \vdash A\end{array} \quad \begin{array}{c}\vdots\ \gamma_2 \\ G \mid \Gamma \vdash B\end{array}}{G \mid B \supset C, \Gamma \vdash A}\ \supset\vdash$$

We apply the induction hypothesis to the left premise and H_2, and apply $(\supset\vdash)$:

$$\frac{G \mid C, \Gamma, \Pi^* \vdash \Lambda \quad G \mid \Gamma \vdash B}{G \mid B \supset C, \Gamma, \Pi^* \vdash \Lambda}$$

(5) Both γ and δ end in logical inferences acting on a cut formula. For instance, if $A = B \supset C$ we have

$$\frac{\begin{array}{c}\vdots\ \gamma_1 \\ G \mid B, \Gamma \vdash C\end{array}}{G \mid \Gamma \vdash B \supset C}\ \vdash\supset \qquad \frac{\begin{array}{c}\vdots\ \delta_1 \\ G \mid \Pi_1 \vdash B\end{array} \quad \begin{array}{c}\vdots\ \delta_2 \\ G \mid C, \Pi_2 \vdash \Lambda\end{array}}{G \mid B \supset C, \Pi_1, \Pi_2 \vdash \Lambda}\ \supset\vdash$$

First we find proofs δ_1' and δ_2' of

$$G \mid \Gamma, \Pi_1^* \vdash B \quad \text{and} \quad G \mid C, \Gamma, \Pi_2^* \vdash \Lambda$$

either by applying the induction hypothesis to γ and δ_1 or δ_2 if Π_1 or Π_2, respectively, contain $B \supset C$, or otherwise by adding (ic)-inferences to δ_1 and δ_2. Now apply the induction hypothesis based on the reduced degree of the cut formulas twice: first to δ_1' and γ_1 to obtain $G \mid \Gamma, \Gamma, \Pi_1^* \vdash C$, and then to the resulting proof and δ_2' to obtain

$$G \mid \Gamma, \Gamma, \Gamma, \Pi_1^*, \Pi_2^* \vdash \Lambda.$$

The desired result follows by several applications of (ic).

The other cases are similar and are left to the reader. □

Cut elimination is a basic prerequisite for proof theoretic and computational treatments of a logic. As an immediate consequence of cut elimination we have the subformula property: every **IF**-valid formula has a proof which only contains subformulas of the endformula (plus possibly propositional variables used in (tt)). Another important corollary is the midhypersequent theorem. It corresponds to Herbrand's Theorem for classical logic and is thus the basis for any resolution-style automated proof method.

Theorem 7. *Any hypersequent H with only prefix formulas has a proof where no propositional inference follows a quantifier inference. Such a proof contains one or more hypersequents M, called midhypersequents, so that M contains no quantifiers, all the inferences above M are propositional or structural, and all the inferences below M are either quantifier inferences of structural inferences.*

Proof. This is proved exactly as for the classical and intuitionistic case (see Takeuti [12]). First, observe that all axioms are cut-free derivable from atomic axioms. The cut-elimination theorem thus provides us with a cut-free proof π of H from atomic axioms. Next, observe that the $(\vee\vdash)$ rule can be simulated without using cuts by the rule

$$\dfrac{G \mid A, \Gamma \vdash \Delta_1 \quad G \mid B, \Gamma \vdash \Delta_2}{G \mid A \vee B, \Gamma \vdash \Delta_1 \mid A \vee B, \Gamma \vdash \Delta_2}\ \vee\vdash'$$

The rule can be derived as follows (we omit side sequents):

$$\dfrac{\dfrac{\dfrac{A,\Gamma\vdash\Delta_1 \quad B,\Gamma\vdash\Delta_2}{B,\Gamma\vdash\Delta_1 \mid A,\Gamma\vdash\Delta_2}\ cm \qquad A,\Gamma\vdash\Delta_1}{A\vee B,\Gamma\vdash\Delta_1\mid A,\Gamma\vdash\Delta_2}\ \vee\vdash \qquad B,\Gamma\vdash\Delta_2}{A\vee B,\Gamma\vdash\Delta_1\mid A\vee B,\Gamma\vdash\Delta_2}\ \vee\vdash$$

Of course, $(\vee\vdash')$ together with (ec) simulates $(\vee\vdash)$. We replace all applications of $(\vee\vdash)$ by applications of $(\vee\vdash')$ in our cut-free proof.

Define the order of a quantifier inference in π to be the number of propositional inferences under it, and the order of π as the sum of the orders of its quantifier inferences. The proof is by induction on the order of π. The only interesting case is of $(\vee\vdash')$ occurring below a quantifier inference, since this case does not work for intuitionistic logic.

Suppose π contains a $(\vdash\forall)$ inference above a $(\vee\vdash')$ inference, and so that all the inferences in between are structural. We have the following situation:

$$\dfrac{\begin{array}{c}\vdots\\ G\mid A,\Gamma\vdash\Delta\end{array} \quad \begin{array}{c}\vdots\ \delta'\\ \dfrac{G'\mid\Gamma'\vdash A(a)}{G'\mid\Gamma'\vdash(\forall x)A(x)}\ \vdash\forall\\ \vdots\ \delta\\ G\mid B,\Gamma\vdash(\forall x)A(x)\end{array}}{G\mid A\vee B,\Gamma\vdash\Delta\mid A\vee B,\Gamma\vdash(\forall x)A(x)}\ \vee\vdash'$$

where δ contains only structural inferences. We reduce the order of π by replacing this part of π by:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ G\mid A,\Gamma\vdash\Delta\end{array} \quad \begin{array}{c}\vdots\ \delta'\\ G'\mid\Gamma'\vdash A(a)\\ \vdots\ \delta\\ G\mid B,\Gamma\vdash A(a)\end{array}}{G\mid A\vee B,\Gamma\vdash\Delta\mid A\vee B,\Gamma\vdash A(a)}\ \vee\vdash'}{G\mid A\vee B,\Gamma\vdash\Delta\mid A\vee B,\Gamma\vdash(\forall x)A(x)}\ \vdash\forall$$

□

5 Elimination of the Takeuti-Titani Rule

The Takeuti-Titani rule is the least understood feature of the original Takeuti-Titani axiomatization of **IF**. We show below that the rule can be eliminated from proofs in **HIF**. This had been posed as a problem by Takano [11]. The proof is by induction on the number of applications of (*tt*) and the length of the proof. The exact complexity of the elimination procedure is still to be investigated. The (*tt*) rule can have significant effects on proof structure. For instance, one of the calculi in Avron [2] uses the split rule

$$\frac{G \mid \Gamma, \Gamma' \vdash \Delta}{G \mid \Gamma \vdash \Delta \mid \Gamma' \vdash \Delta}\ split$$

If this rule is added to **HIF**, it is possible to transform proofs so that each application of the communication rule has a premise which is a propositional axiom. This is not possible without (*tt*). The transformation works by replacing each occurrence of the communication rule by

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{q \vdash q \quad \dfrac{p \vdash p \quad \dfrac{G_1 \mid \Gamma_1, \Gamma_1' \vdash A_1}{G_1 \mid \Gamma_1 \vdash A_1 \mid \Gamma_1' \vdash A_1}\ split}{G_1 \mid \Gamma_1 \vdash A_1 \vdash \Gamma_1' \vdash p \mid p \vdash A_1}\ cm}{G_1 \mid \Gamma_1 \vdash A_1 \mid \Gamma_1' \vdash q \mid p \vdash A_1 \mid q \vdash p}\ cm \quad \dfrac{\dfrac{G_2 \mid \Gamma_2, \Gamma_2' \vdash A_2}{G_2 \mid \Gamma_2 \vdash A_2 \mid \Gamma_2' \vdash A_2}\ split \quad q \vdash q}{G_2 \mid \Gamma_2 \vdash q \mid q \vdash A_2 \mid \Gamma_2' \vdash A_2}\ cm}{G_1 \mid G_2 \mid \Gamma_1 \vdash A_1 \mid \Gamma_2 \vdash q \mid p \vdash A_1 \mid \Gamma_2 \vdash p \mid q \vdash A_2 \mid \Gamma_2' \vdash A_2}\ cut}{G_1 \mid G_2 \mid \Gamma_1 \vdash A_1 \mid \Gamma_2 \vdash A_2 \mid p \vdash A_1 \mid \Gamma_2 \vdash p \mid \Gamma_2' \vdash A_2}\ tt}{G_1 \mid G_2 \mid \Gamma_1 \vdash A_1 \mid \Gamma_2 \vdash A_2 \mid \Gamma_2 \vdash A_1 \mid \Gamma_2' \vdash A_2}\ tt}{G_1 \mid G_2 \mid \Gamma_1, \Gamma_2' \vdash A_1 \mid \Gamma_1', \Gamma_2 \vdash A_2}$$

Proposition 8. *Let δ be a* **HIF****-derivation of hypersequent H with length k, where H is of the form*

$$G \mid \Gamma_1, \Pi_1 \vdash \Delta_1, \Pi_1' \mid \ldots \mid \Gamma_n, \Pi_n \vdash \Delta_n, \Pi_n'$$

and $\bigcup \Pi_i \subseteq \{p\}$, $\Pi_i' = \emptyset$, and p does not occur in G, Γ_i or Δ_i ($\bigcup \Pi_i' = \{p\}$, $\Pi_i = \emptyset$, and p does not occur in G, Γ_i or Δ_i).

Then the hypersequent $G \mid \Gamma_{i_1} \vdash \Delta_{i_1} \mid \ldots \mid \Gamma_{i_m} \vdash \Delta_{i_m}$ is derivable in length $\leq k$.

Proof. Easy induction on k. Every occurrence of p must arise from a weakening, simply delete all these weakenings.

Theorem 9. *Applications of (tt) can be eliminated from* **HIF***-derivations.*

This follows from the following lemma by induction on the number of applications of (*tt*) in a given **HIF**$^-$-derivation.

Lemma 10. *If δ is an* **HIF****-derivation of*

$$H = G \mid \Phi_1 \vdash \Pi_1 \mid \dots \Phi_n \vdash \Pi_n \mid \Pi'_1, \Psi_1 \vdash \Sigma_1 \mid \dots \mid \Pi'_m, \Psi_m \vdash \Sigma_m,$$

where p does not occur in G, Φ_i, Ψ_i or Σ_i, and $\bigcup \Pi_i \cup \bigcup \Pi'_i \subseteq \{p\}$, then there is a **HIF****-derivation of*

$$H^* = G \mid \Phi_1, \dots, \Phi_n, \Psi_1 \vdash \Sigma_1 \mid \dots \mid \Phi_1, \dots, \Phi_n, \Psi_m \vdash \Sigma_m.$$

Proof. By induction on the length of δ. We distinguish cases according to the last inference I in δ. For simplicity, we will write p in what follows below instead of Π_i or Π'_i with the understanding that it denotes an arbitrary multiset of p's.

(1) The conclusion of of I is so that p only occurs on the right side of sequents, or only on the left side. Then Prop. 8 applies, and the desired hypersequent can be derived without (tt).

(2) I applies to sequents in G. Then the induction hypothesis can be applied to the premise(s) of I and appropriate inferences added below.

(3) I is structural inference other than (cut) and (cm), or a logical inference with only one premise, or a logical inference which applies to a Σ_i. These cases are likewise handled in an obvious manner and are unproblematic. One instructive example might be the case of $(\supset\vdash)$. Here the premises would be of the form, say,

$$G \mid \Phi_1 \vdash p \mid \Phi_2 \vdash p \dots \mid \Phi_n \vdash p \mid p, \Psi_1 \vdash \Sigma_1 \mid \dots \mid p, \Psi_m \vdash \Sigma_m \mid p, \Gamma_1 \vdash A$$

$$G \mid \Phi_1 \vdash p \mid \Phi_2 \vdash p \dots \mid \Phi_n \vdash p \mid p, \Psi_1 \vdash \Sigma_1 \mid \dots \mid p, \Psi_m \vdash \Sigma_m \mid B, \Gamma_2 \vdash p$$

Let $\Phi = \Phi_1, \dots, \Phi_n$. The induction hypothesis provides us with

$$G \mid \Phi, \Psi_1 \vdash \Sigma_1 \mid \dots \mid \Phi, \Psi_m \vdash \Sigma_m \mid \Phi, \Gamma_1 \vdash A$$

$$G \mid B, \Gamma_2, \Phi, \Psi_1 \vdash \Sigma_1 \mid \dots \mid B, \Gamma_2, \Phi, \Psi_m \vdash \Sigma_m$$

We obtain the desired hypersequent by applying $(\supset\vdash)$ successively m times, together with some contractions.

(4) I is a cut. There are several cases to consider, most of which are routine. The only tricky case is when the cut formula is p and p occurs both on the left and the right side of sequents in both premises of the cut. For simplicity, let us consider the cut rule in its multiplicate formulation

$$G \mid \Phi_1 \vdash p \mid \dots \mid \Phi_n \vdash p \mid p, \Psi_1 \vdash \Sigma_1 \mid \dots \mid p, \Psi_m \vdash \Sigma_m \mid \Gamma \vdash p$$

$$G \mid \Phi_1 \vdash p \mid \dots \mid \Phi_n \vdash p \mid p, \Psi_1 \vdash \Sigma_1 \mid \dots \mid p, \Psi_m \vdash \Sigma_m \mid p, \Pi \vdash \Lambda$$

We want to find a derivation of

$$G \mid \Phi, \Psi_1 \vdash \Sigma_1 \mid \Phi, \Psi_m \vdash \Sigma_m \mid \Gamma, \Pi \vdash \Lambda$$

where $\Phi = \Phi_1, \dots, \Phi_n$. The induction hypothesis applied to the premises of the cut gives us

$$G \mid \Gamma, \Phi, \Psi_1 \vdash \Sigma_1 \mid \dots \mid \Gamma, \Phi, \Psi_m \vdash \Sigma_m$$

$$G \mid \Phi, \Psi_1 \vdash \Sigma_1 \mid \dots \mid \Phi, \Psi_m \vdash \Sigma_m \mid \Phi, \Pi \vdash \Lambda$$

We obtain the desired hypersequent by m successive applications of (cm).

(5) I is $(\vee\vdash)$, or $(\exists\vdash)$ applying to Φ_i or Ψ_i. Consider the case of $(\vee\vdash)$, the others are treated similarly. The premises of I are, for example,

$$G \mid A, \Phi_1 \vdash p \mid \Phi_2 \vdash p \ldots \mid \Phi_n \vdash p \mid p, \Psi_1 \vdash \Sigma_1 \mid \ldots \mid p, \Psi_m \vdash \Sigma_m$$
$$G \mid B, \Phi_1 \vdash p \mid \Phi_2 \vdash p \ldots \mid \Phi_n \vdash p \mid p, \Psi_1 \vdash \Sigma_1 \mid \ldots \mid p, \Psi_m \vdash \Sigma_m$$

By induction hypothesis, we obtain

$$G \mid A, \Phi_1, \ldots, \Phi_n, \Psi_1 \vdash \Sigma_1 \mid \ldots \mid A, \Phi_1, \ldots, \Phi_n, \Psi_m \vdash \Sigma_m$$
$$G \mid B, \Phi_1, \ldots, \Phi_n, \Psi_1 \vdash \Sigma_1 \mid \ldots \mid B, \Phi_1, \ldots, \Phi_n, \Psi_m \vdash \Sigma_m$$

It is not straightforwardly possible to derive the desired hypersequent from these. If $\Psi_i = \{P_{i1}, \ldots, P_{ik_i}\}$, let $Q_i = P_{i1} \supset \ldots P_{ik_i} \supset \Sigma_i$. Then we do easily obtain, however, the following by repeated application of $(\vdash\supset)$, $(\vdash\vee)$ and (ec):

$$G \mid A, \Phi_1, \ldots, \Phi_n \vdash Q_1 \vee \ldots \vee Q_m$$
$$G \mid B, \Phi_1, \ldots, \Phi_n \vdash Q_1 \vee \ldots \vee Q_m$$

Now a single application of $(\vee\vdash)$, plus (ec) gives us

$$K = G \mid \underbrace{A \vee B, \Phi_1, \ldots, \Phi_n}_{\Gamma} \vdash Q_1 \vee \ldots \vee Q_m$$

Then we derive, using $m-1$ cuts:

$$\dfrac{\dfrac{K \quad \overset{\vdots\ \delta_1}{Q_1 \vee Q \vdash Q_1 \mid Q_1 \vee Q \vdash Q}}{\Gamma \vdash Q_1 \mid \Gamma \vdash \underbrace{Q_2 \vee \ldots \vee Q_m}_{Q}}}{}$$

$$\dfrac{\overset{\vdots}{\Gamma \vdash Q_1 \mid \ldots \mid \Gamma \vdash Q_{m-1} \vee Q_m} \quad \overset{\vdots\ \delta_{m-1}}{Q_{m-1} \vee Q_m \vdash Q_{m-1} \mid Q_{m-1} \vee Q_m \vdash Q_m}}{\Gamma \vdash Q_1 \mid \ldots \mid \Gamma \vdash Q_m}$$

where δ_i is the derivation

$$\dfrac{Q_i \vdash Q_i \qquad \dfrac{\dfrac{Q \vdash Q \quad Q_i \vdash Q_i}{Q \vdash Q_i \mid Q_i \vdash Q}\,cm \qquad Q \vdash Q}{Q \vdash Q_i \mid Q_i \vee Q \vdash Q}\,\vee\vdash}{Q_i \vee \underbrace{Q_{i+1} \vee \ldots \vee Q_m}_{Q} \vdash Q_i \mid Q_i \vee \ldots \vee Q_m \vdash \underbrace{Q_{i+1} \vee \ldots \vee Q_m}_{Q}}\,\vee\vdash$$

The desired hypersequent is obtained by m cuts with

$$Q_i, P_{i1}, \ldots, P_{ik_i} \vdash \Sigma_i$$

(6) I is a communication rule. This is the most involved case, as several subcases have to be distinguished according to which of the two communicated sequents contains p. Neither of these cases are problematic. We present two examples:

(a) One of the communicated sequents contains p on the right. Then the premises of I are

$$G \mid \Phi_1 \vdash p \mid \ldots \mid \Phi_n \vdash p \mid p, \Psi_1 \vdash \Sigma_1 \mid \ldots \mid p, \Psi_m \vdash \Sigma_m \mid \Theta_1, \Theta_1' \vdash p$$
$$G \mid \Phi_1 \vdash p \mid \ldots \mid \Phi_n \vdash p \mid p, \Psi_1 \vdash \Sigma_1 \mid \ldots \mid p, \Psi_m \vdash \Sigma_m \mid \Theta_2, \Theta_2' \vdash \Xi_2$$

where. The induction hypothesis applies to these two hypersequents. If we write $\Phi = \Phi_1, \ldots, \Phi_n$, we have

$$G \mid \boxed{\Theta_1, \Theta_1', \Phi, \Psi_1 \vdash \Sigma_1} \mid \ldots \mid \Theta_1, \Theta_1', \Phi, \Psi_m \vdash \Sigma_m$$
$$G \mid \boxed{\Theta_2, \Theta_2' \vdash \Xi} \mid \Phi, \Psi_1 \vdash \Sigma_1 \mid \ldots \mid \Phi, \Psi_m \vdash \Sigma_m$$

We obtain the desired result by applying m instances of (cm), internal weakenings and external contractions as necessary, to obtain, in sequence

$$G \mid \Theta_1, \Theta_2', \Phi, \Psi_1 \vdash \Sigma_1 \mid \ldots \mid \Theta_1, \Theta_1', \Phi, \Psi_m \vdash \Sigma_m \mid \Theta_1', \Theta_2 \vdash \Xi$$
$$\ddots$$
$$G \mid \Theta_1, \Theta_2', \Phi, \Psi_1 \vdash \Sigma_1 \mid \ldots \mid \Theta_1, \Theta_2', \Phi, \Psi_m \vdash \Sigma_m \mid \Theta_1', \Theta_2 \vdash \Xi$$

The sequents participating in the application of (cm) are marked by boxes. The original end hypersequent follows from the last one by internal weakenings.

(b) The communicated sequents both contain p, once on the right, once on the left. The premises of I are

$$G \mid \Phi_1 \vdash p \mid \ldots \mid \Phi_n \vdash p \mid p, \Psi_1 \vdash \Sigma_1 \mid \ldots \mid p, \Psi_m \vdash \Sigma_m \mid \Theta_1, \Theta_1' \vdash p$$
$$G \mid \Phi_1 \vdash p \mid \ldots \mid \Phi_n \vdash p \mid p, \Psi_1 \vdash \Sigma_1 \mid \ldots \mid p, \Psi_m \vdash \Sigma_m \mid p, \Theta_2, \Theta_2' \vdash \Xi$$

We have proofs of

$$G \mid \Theta_1, \Theta_1', \Phi, \Psi_1 \vdash \Sigma_1 \mid \ldots \mid \Theta_1, \Theta_1', \Phi, \Psi_m \vdash \Sigma_m$$
$$G \mid \Phi, \Psi_1 \vdash \Sigma_1 \mid \ldots \mid \Phi, \Psi_m \vdash \Sigma_m \mid \Theta_2, \Theta_2', \Phi \vdash \Xi$$

Again, a sequence of m applications of (cm), together with internal weakenings and external contractions produces the desired end sequent. □

Note that in case (5), several new cuts are introduced. As a consequence, the elimination procedure does not directly work for cut-free proofs. If a proof with neither cut nor communication is required, the elimination procedure has to be combined with the cut-elimination procedure of Thm. 5. The additional cuts can be avoided by replacing $(\vee\vdash)$ and $(\exists\vdash)$ by the following generalized rules:

$$\frac{G \mid A, \Gamma_1 \vdash \Delta_1 \mid \ldots \mid A, \Gamma_n \vdash \Delta_n \quad G \mid B, \Gamma_1 \vdash \Delta_1 \mid \ldots \mid B, \Gamma_n \vdash \Delta_n}{G \mid A \vee B, \Gamma_1 \vdash \Delta_1 \mid \ldots \mid A \vee B, \Gamma_n \vdash \Delta_n} \vee\vdash^*$$

$$\frac{G \mid A(a), \Gamma_1 \vdash \Delta_1 \mid \ldots \mid A(a), \Gamma_n \vdash \Delta_n}{G \mid (\exists x)A(x), \Gamma_1 \vdash \Delta_1 \mid \ldots \mid (\exists x)A(x), \Gamma_n \vdash \Delta_n} \exists\vdash^*$$

These rules, however, cannot be simulated by the ordinary rules without using cut (the simulation with cut is given in case (5)). By changing case (5) accordingly, the elimination procedure will transform a cut-free **HIF**-derivation into a cut-free one without (tt), but with $(\vee\vdash^*)$ and $(\exists\vdash^*)$.

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