A Paedagogic Example of Cut-Elimination

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1 Introduction

In his groundbreaking 1934 paper, "Untersuchungen über das logische Schließen" [1934], Gerhard Genzen introduced a new calculus for first-order logic, the so-called *sequent calculus* **LK**. In contrast to the then (and now) predominant Hilbert-style calculi, **LK**'s power lies in its rules, not in its axioms. *Sequents* are expressions of the form

$$\Gamma \to \Pi,$$

where Π and Γ are lists of predicate formulas, '$\to$' is called the *sequent arrow*. For someone used to the Hilbert-style calculus, where one only works with single formulas, trying to deduce the end formula from a list of axioms by only two rules, namely *modus ponens* and *generalization*, this seems a rather strange formulation of logic. Indeed it springs from a very deep insight of Genzen's: he was aware that classical logic is symmetric in the truth values *true*and *false*. In a sequent, the formulas to the left of the sequent arrow contain something false, or those to the right contain something true. In fact, it is a straightforward task to extend this idea (and the logical calculus) to multi-valued logics by introducing more than two places for formulas to stand in a sequent. One could also interpret a sequent as an implication: the conjunction of the formulas on the left implies the disjunction of the formulas on the right. To prove a formula F in sequent calculus, one tries to deduce the sequent $\to F$.

The axioms of **LK** are simply sequents of the form $F \to F$, meaning: either F is true or F is false. For every logical symbol one has introduction rules for the left side and the right side of the sequent, one has structural rules for reordering formulas in the sequent etc., and one special rule, the *cut rule*:

$$\frac{\Gamma \to \Delta, A \quad A, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda}\ \text{cut}$$

One of the most important facts about **LK** is the Cut-Elimination Theorem, also known as Gentzen's *Hauptsatz*: if a formula is provable in **LK**, then it is provable in **LK** without a cut.

The Cut-Elimination Theorem is one of the most important results of proof theory. It was first proved by Gentzen [1934] and has a number of interesting consequences, e.g., Craig's Interpolation Theorem, and Beth's Definability Lemma [1980]. Furthermore, Gentzen's proof is constructive and gives a proof-theoretic *algorithm*. In contrast to other well-known and important results in mathematical logic, e.g., Gödel's Incompleteness Theorems, the Cut-Elimination Theorem is still little understood in the mathematical community. Presumably one reason for this is that there are no non-logical examples for cut-elimination as there are examples illustrating the content of the Incompleteness results (see, e.g., [1977]).

We present here such a motivating example in the theory of lattices. It is well known that there is a one-one correspondence between lattices as equationally defined algebras (by the laws of commutativity, associativity, idempotency, and absorption), and partial orders where there exist infimum and supremum for every two elements x and y of the domain [1978]. The meet of two elements then equals their infimum, and their join equals their supremum. We introduce a calculus similar to Gentzen's **LK** for proving inequalities in lattices, i.e., expressions of the form

$$X \leq Y,$$

where X and Y are expressions containing variables ranging over elements of the lattice, and the symbols $\cup$ and $\cap$ for the join and meet in a lattice. We denote the theory containing all such inequalities which are true in every lattice, and—*par abus de langage*—also our calculus by **LI**. This calculus contains the rule *trans* formalizing transitivity in the partial order, which is the analogon to the cut rule in **LK**; we give two proofs that this rule is in fact redundant.

2 Preliminaries

First, we shall make precise the language and formal system we are working with:

Definition 1 The set of *symbols* of **LI** consist of the following:

1. variables: x_0, x_1, x_2, $\ldots$,
2. relation symbols: $\cap$, $\cup$, $\leq$, and
3. auxiliary symbols: (and).

The set of *terms* of **LI** is the least set closed under the following formation rules:

1. Every variable is a term.
2. If X and Y are terms, then $(X \cup Y)$ and $(X \cap Y)$ are terms.

The set of *formulas* of **LI** is the set of all expressions of the form $X \leq Y$, where X and Y are terms.

In the following, we shall use lowercase letters for denoting variables, and uppercase letters for denoting terms, and we shall sometimes omit the outermost parentheses in terms.

We shall not go into much detail about the model theory of LI, as it is quite obvious: A *structure* for **LI** is a lattice $\langle L, \cap, \cup \rangle$; an *interpretation* of a formula F is a structure L together with a mapping $\varphi : \mathrm{VAR}(F) \to L$ that assigns an element of L to every variable in F. Every such mapping can uniquely be extended to a homomorphism $\overline{\varphi}$ from the term algebra over $\mathrm{VAR}(F)$ to L. If $F \equiv X \leq Y$, then F is said to be *true* in L under φ, if $\overline{\varphi}(X) \leq \overline{\varphi}(Y)$ in the induced partial order on L, otherwise it is called *false*. An interpretation that makes F true, or *verifies* it, is called a *model* of F, and an interpretation which makes F false is called a *countermodel* or *counterexample* for F. If every interpretation for F is a model, then we say that F is *valid*.

We now introduce the following calculus for **LI**:

1. Axioms: all formulas of the form: (x is a variable)

$$x \leq x$$

2. Rules: (X, Y, and Z are terms)

$$\frac{X \leq Z}{(X \cap Y) \leq Z}\ \cap\text{:left} \qquad \frac{X \leq Y \quad X \leq Z}{X \leq (Y \cap Z)}\ \cap\text{:right}$$

$$\frac{X \leq Z \quad Y \leq Z}{(X \cup Y) \leq Z}\ \cup\text{:left} \qquad \frac{X \leq Y}{X \leq (Y \cup Z)}\ \cup\text{:right}$$

$$\frac{(X \cap Y) \leq Z}{(Y \cap X) \leq Z}\ \text{comm:left} \qquad \frac{X \leq (Y \cup Z)}{X \leq (Z \cup Y)}\ \text{comm:right}$$

$$\frac{X \leq Z \quad Z \leq Y}{X \leq Y}\ \text{trans}$$

A formal *proof* of a formula in **LI** is a tree where every node is labelled with a formula of **LI**: every leaf is labelled with an axiom, and the successors of a node that is labelled with an instance of the consequent of a rule are labelled with instances of the antecedents of the same rule. A formula F is called *provable* iff it has a proof.

The calculus is in fact *sound*, i.e., it produces only valid formulas, and *complete*, i.e., every valid formula of **LI** is provable in it, but we shall postpone the proof of these facts until the next section.

Example 1 *The following example is the proof of a weak distributivity law, which holds in every lattice:*

$$\dfrac{\dfrac{\dfrac{x \leq x}{x \cap y \leq x} \quad \dfrac{\dfrac{y \leq y}{x \cap y \leq y}}{x \cap y \leq y \cup z}}{x \cap y \leq x \cap (y \cup z)} \quad \dfrac{\dfrac{x \leq x}{x \cap z \leq x} \quad \dfrac{\dfrac{z \leq z}{x \cap z \leq z}}{x \cap z \leq y \cup z}}{x \cap z \leq (x \cap (y \cup z))}}{(x \cap y) \cup (x \cap z) \leq x \cap (y \cup z)}$$

Example 2 *The following two proofs illustrate the use of the rule trans:*

$$\dfrac{\dfrac{x \leq x}{x \leq x \cup y}}{x \cap y \leq x \cup y} \qquad \dfrac{\dfrac{x \leq x}{x \cap y \leq x} \quad \dfrac{x \leq x}{x \leq x \cup y}}{x \cap y \leq x \cup y}$$

3 Elimination of Cuts

The cut-elimination theorem for **LK** is the statement that every formula F provable in first-order logic is also provable without using the cut rule. Gentzen's proof is constructive in the sense that it gives a procedure for *directly transforming* a proof of F using the cut rule to a proof without cut. The proof works by defining two measures on cuts in proofs: the *rank* of a cut is the maximum distance between the cut and the first occurence of the cut formula in the proof tree above the cut. The *grade* of a cut is the complexity of the cut formula. One then proves the theorem by showing that from a proof containing a single cut, this cut can be eliminated; this is done by double induction on the rank and grade of the cut.

The cut-elimination theorem is also a corollary to the completeness theorem as proved by methods of Schütte: He showed that **LK** without the cut is complete, and hence, since **LK** is sound, that the cut rule is a derived rule of **LK**. To prove completeness one shows that, for every formula F of first-order logic, there either is a proof or a counterexample for F. Since in every rule in **LK** other than cut, the antecedents are less complex than the consequents, one can construct a proof tree for F "backwards". If F is valid, this procedure stops with all leaves of the proof tree being axioms—then we have obtained a proof for F. Otherwise, we can construct a countermodel for F from the tree. For first-order formulas F this tree is in general infinite, but for propositional formulas these trees are finite, and thus Schütte's method also gives a decision procedure for propositional logic.

In the following two sections, we translate the proofs of Gentzen and Schütte into the calculus **LI** and prove the analogon to Gentzen's *Hauptsatz* for **LI**. We use *cut* to mean *application of the rule trans*.

4 Elimination of Transitivity Gentzen-style

Theorem 1 *A proof of a formula F in **LI** can be effectively transformed into a proof of F without cuts, i.e., a proof of F in **LI** without trans.*

Proof. We observe that, if we could eliminate cuts from proofs containing only one cut as the last inference, then we can eliminate cuts from all proofs: simply eliminate the topmost cuts in the proof tree until there is only one cut left. Thus it suffices to show the cut-elimination theorem for such proofs.

Let Y denote the 'cut formula' in the proof considered:

$$\dfrac{\begin{matrix}\vdots \\ X \leq Y\end{matrix} \quad \begin{matrix}\vdots \\ Y \leq Z\end{matrix}}{X \leq Z}$$

Let the *rank* of a cut be the height of the subproof ending with the cut, and the *left* and *right rank* be the height of the subproof ending with the left or right antecedent of the cut, respectively, plus 1. The rank of a cut then equals the maximum of its left and right rank. We prove the theorem by induction on the rank n of the cut.

1. If $n = 1$, then $Y \equiv x$ and we have an inference from two axioms:

   $$\dfrac{x \leq x \quad x \leq x}{x \leq x}$$

   which can be replaced by the axiom

   $$x \leq x$$

   and is thus provable without cut.

2. If $n > 1$ we distinguish cases according to the form of Y:

   (a) $Y \equiv y$, y a variable: We transform the proof according to the following rules

   i. X or Z is a variable, say $X \equiv x$. We have

   $$\dfrac{x \leq y \quad y \leq Z}{x \leq Z}$$

   Since $x \leq y$ is only derivable if x and y are the same variables, we have $x \equiv y$ and as a proof the right upper branch of the cut

   $$\begin{matrix}\vdots \\ x \leq Z\end{matrix}$$

   which is of height $n - 1$ and contains no cut.

ii. $X \equiv A \cap B$ and the last inference on the left branch was $\cap$:left or comm:left. The proof then looks like:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A \leq Y\end{matrix}}{A \cap B \leq Y} \quad \begin{matrix}\vdots\\ Y \leq Z\end{matrix}}{A \cap B \leq Z} \qquad\qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ B \cap A \leq Y\end{matrix}}{A \cap B \leq Y} \quad \begin{matrix}\vdots\\ Y \leq Z\end{matrix}}{A \cap B \leq Z}$$

We can move these inferences below the cut. Then the left rank of the resulting cut is at most $n-1$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A \leq Y\end{matrix} \quad \begin{matrix}\vdots\\ Y \leq Z\end{matrix}}{A \leq Z}}{A \cap B \leq Z} \qquad\qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ B \cap A \leq Y\end{matrix} \quad \begin{matrix}\vdots\\ Y \leq Z\end{matrix}}{B \cap A \leq Z}}{A \cap B \leq Z}$$

iii. $Z \equiv A \cup B$ and the last inference on the right branch was $\cup$:right or comm:right. Similarly. We obtain a proof whose only cut has right rank at most $n-1$.

iv. $X \equiv A \cup B$ and the last inference on the left branch was $\cup$:left. The proof then looks like:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A \leq Y\end{matrix} \quad \begin{matrix}\vdots\\ B \leq Y\end{matrix}}{A \cup B \leq Y} \quad \begin{matrix}\vdots\\ Y \leq Z\end{matrix}}{A \cup B \leq Z}$$

We copy the subproof ending in $Y \leq Z$ and transform the proof as follows:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A \leq Y\end{matrix} \quad \begin{matrix}\vdots\\ Y \leq Z\end{matrix}}{A \leq Z} \quad \dfrac{\begin{matrix}\vdots\\ B \leq Y\end{matrix} \quad \begin{matrix}\vdots\\ Y \leq Z\end{matrix}}{B \leq Z}}{A \cup B \leq Z}$$

In this proof, the left ranks of the two occuring cuts are both at most $n-1$.

v. $Z \equiv A \cap B$ and the last inference on the right branch was $\cap$:right. Similarly. We obtain a proof where all cuts have right rank at most $n-1$.

By transforming a proof ending with a cut according to (i)–(v) as above, and in this order, we obtain a proof of F containing cuts whose ranks are all less than n, the rank of the original cut: (ii) and (iv) reduce the left ranks of all cuts by at least one, (iii) and (v) the right ranks by at least one, and the *ranks* of all resulting cuts are at most n. Hence, by induction hypothesis, and the remarks made at the beginning of the proof, the cuts can be eliminated from the transformed proof.

(b) $Y \equiv (U \cup V)$. We can assume that $\cup$:right and $\cup$:left were the last rules applied before trans, since otherwise we can transform the proof to one where this condition holds of every subproof ending with a cut by applying (ii)–(v) as above. The last lines of the proof have the form:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ X \leq U\end{matrix}}{X \leq U \cup V} \quad \dfrac{\begin{matrix}\vdots\\ U \leq Z\end{matrix} \quad \begin{matrix}\vdots\\ V \leq Z\end{matrix}}{U \cup V \leq Z}}{X \leq Z}$$

which can be transformed into a proof of height $n-1$ ending in a cut:

$$\dfrac{\begin{matrix}\vdots & \vdots \\ X \leq U & U \leq Z\end{matrix}}{X \leq Z}$$

Again, this cut can be eliminated by induction hypothesis.

(c) $Y \equiv U \cap V$. Similarly.

NB. If one looks close enough, one sees that the above proof is actually a double induction proof on the rank of the cut, and the *grade of of the cut formula*, i.e., its term complexity: (a) treats the base case for the induction on the complexity of Y, (b) and (c) the induction step. As already stated, this is also the way the Gentzen originally proved his *Hauptsatz*.

5 Completeness of LI Schütte-style

One standard way to prove that a rule is a derived rule of some formal system, is to prove that the system *with* the rule is *sound*, i.e., only proves valid formulas, and that the system *without* the rule is *complete*, i.e., that every valid formula can be proved in it. It is obvious that trans is a sound rule, so if we succeed in proving **LI** to be complete, we also have shown the redundancy of trans.

Theorem 2 ***LI*** *is sound, i.e., every formula derivable in* ***LI*** *holds in every lattice.*

Proof. Obviously, by the reflexivity of the partial order, the axioms are true in every lattice. Since the meet and join of two lattice elements x and y are lower and upper bounds of $\{x, y\}$, respectively, $x \cap y$ is a lower bound of $\{x\}$ and $x \cup y$ an upper bound of $\{y\}$. By transitivity, the soundness of $\cap$:left and $\cup$:right follows. If $x \leq y$ and $x \leq z$ (i.e., x is a lower bound of $\{y, z\}$), then by the definition of greatest lower bound, x is also less than $y \cap z$ which means that $\cap$:right is sound. Similarly for $\cup$:left. The rules comm:left and comm:right are evidently sound, since meet and join are commutative. The cut rule, trans, is simply a formalization of transitivity and thus sound as well.

It is intuitively clear that **LI** is complete, since we have everything we know about lattices somehow coded: we have reflexivity as the axioms, commutativity as two rules, we get associativity by rearranging the proof tree, we have formalized that meet and join are greatest lower and least upper bound, respectively, and we have transitivity in the form of our cut rule trans. We do not need antisymmetry, since we are not dealing with equality, but of course, if **LI** is complete, we can invent a new calculus proving equalities in lattices by adding a new rule

$$\dfrac{X \leq Y \quad Y \leq X}{X = Y}\ \text{equ}\ .$$

It is probably not so clear that we do not need transitivity for proving all the valid formulas of **LI**. In particular it might seem possible that there exist obscure terms X and Z, denoting elements x and z of a lattice L, with $x \leq z$, but to prove this we have to know about some 'intermediate' element y of L, so that we can use trans to conclude $x \leq z$ from $x \leq y$ and $y \leq z$. The cut-elimination theorem for **LI** states that this is not the case, and this is also the analogon to saying that a formula can be proved 'without detours' in **LK**.

We now go on to show by model-theoretic means that **LI** without trans is complete. We then have, together with the last theorem, that trans is a derived rule of **LI**.

Theorem 3 ***LI*** *is complete, i.e., every true formula of* ***LI*** *is derivable.*

Proof. For reasons of simplicity we will use in the following a slightly modified calculus **LI′**, which is **LI** without the rule trans, and with comm:left and comm:right replaced by the following rules:

$$\frac{Y \leq Z}{(X \cap Y) \leq Z}\ \cap\text{:left}' \qquad\qquad \frac{X \leq Z}{X \leq (Y \cup Z)}\ \cup\text{:right}'.$$

It is obvious that **LI** and **LI′** are equivalent.

Let F be a formula of **LI**. We prove that either we can find a proof of F in **LI′**, in which case F is true and provable, or we can provide a countermodel for F. We do this by building a so-called *reduction tree* for F, which is a finite tree P of **LI**-formulas such that (1) F is at the root of P and (2) every leaf of P is an expression of the form $x \leq y$, x and y variables. From this tree we can either construct a proof of F or prove the existence of a countermodel. The reduction tree looks like a proof tree obtained by applying the rules of **LI′** 'backwards', only that at a stage where either ∩:left or ∩:left′ can be applied, we apply *both* and add the results as the left and right antecedent to P, respectively; similarly for ∪:right and ∪:right′. The function **P** recursively constructs P:

$$\mathbf{P}(F) = \begin{cases} x \leq y & \text{if } F \equiv x \leq y \text{ with} \\ & x,\ y \text{ variables (1)} \\ \dfrac{\mathbf{P}(A \leq Z) \quad \mathbf{P}(B \leq Z)}{A \cup B \leq Z} & \text{if } F \equiv A \cup B \leq Z \text{ (2)} \\ \dfrac{\mathbf{P}(X \leq A) \quad \mathbf{P}(X \leq B)}{X \leq A \cap B} & \text{if } F \equiv X \leq A \cap B \text{ (3)} \\ \dfrac{\mathbf{P}(A \leq Z) \quad ; \quad \mathbf{P}(B \leq Z}{A \cap B \leq Z} & \text{if } F \equiv A \cap B \leq Z \text{ (4)} \\ \dfrac{\mathbf{P}(X \leq A) \quad ; \quad \mathbf{P}(X \leq B)}{X \leq A \cup B} & \text{if } F \equiv X \leq A \cup B \text{ (5)} \end{cases}$$

The order of the cases is important, at each step, the first case in the definition of **P** must be applied for which the condition on the right holds. Since the term complexity decreases with each recursive call of **P**, this procedure terminates and gives us, for every F, exactly one reduction tree P. If, in the process of constructing P, one of the cases (4) or (5) is applied, we call the inference constructed a *choice point*, it represents a choice of whether ∩:left or ∩:left′ (or ∪:right or ∪:right′) should be applied at this point in the proof. A *reduction branch* of the reduction tree P is a proof tree of **LI′** obtained from P by (1) writing the end formula F at the bottom of the tree, (2) at a choice point, choosing one of the two possible proofs and (3) constructing reduction branches of both partial trees above every other inference. Obviously, there are only finitely many reduction branches of every reduction tree (2^c, c the number choice points, to be exact). Here is an example of a reduction tree, with an reduction branch in boldface type, which also defines a proof tree:

$$\dfrac{\dfrac{\boldsymbol{x \leq x} \quad ; \quad y \leq x}{\boldsymbol{x \cap y \leq x}} \qquad \dfrac{\dfrac{\boldsymbol{x \leq x} \quad ; \quad x \leq y}{\boldsymbol{x \leq x \cup y}} \quad ; \quad \dfrac{y \leq x \quad ; \quad y \leq y}{x \leq x \cup y}}{\boldsymbol{x \cap y \leq x \cup y}}}{\boldsymbol{x \cap y \leq x \cap (x \cup y)}}$$

Now there may be a reduction branch R of P which is also a proof of F, i.e., every topmost formula of R is an axiom; we call a reduction branch with this property *closed* and *open* otherwise. If there is a closed reduction branch, we have succeeded in proving that F is valid and found a formal proof in **LI**′ (which can be routinely transformed to a proof in **LI**). Otherwise, no reduction branch of P is a proof; this is the case iff in every reduction branch there is at least one topmost formula of the form $x \leq y$ where x and y are distinct variables.

We follow the definitions of [1968,ch. 1 & 4]. Let $\mathbb{L}$ denote the variety of lattices. $\Theta_{\mathbb{L}}$ is the fully invariant congruence relation of the polynomial algebra $\mathcal{P}^{\omega}(\mathbb{L})$ defined by

$$p \equiv q\ (\Theta_{\mathbb{L}}) \iff p(a_0, \ldots) = q(a_0, \ldots) forall a_i \in L, L \in \mathbb{L}.$$

The factor algebra $\mathcal{P}^{\omega}(\mathbb{L})$ is isomorphic to the free lattice over ω free generators $\mathcal{F}_{\mathbb{L}}(\omega)$. Let the set of *subterms* of $F \equiv X \leq Y$, denoted SUB(F), be the set of all subterms of X and Y from which they were constructed according to the formation rules of definition 1. We give an interpretation for F by a lattice $\langle L, \sqcap, \sqcup \rangle$ and an interpretation function φ:

1. $L := \mathrm{SUB}(F)/\Theta_{\mathbb{L}} \cup \{0, 1\}$
2. $x \sqcap y := \begin{cases} [x \cap y]\Theta_{\mathbb{L}} & \text{if } (x \cap y) \in \mathrm{SUB}(F) \\ 0 & \text{otherwise} \end{cases}$
3. $x \sqcup y := \begin{cases} [x \cup y]\Theta_{\mathbb{L}} & \text{if } (x \cup y) \in \mathrm{SUB}(F) \\ 1 & \text{otherwise} \end{cases}$
4. $\varphi : \mathrm{VAR}(F) \rightarrow L, \varphi(x) := x$

Note that in L, the members of VAR(F) are all mutually incomparable, so $x \leq y$ holds in L iff $x \equiv y$ for variables x and y. We prove the theorem for formulas $F \equiv X \leq Y$ first, where $\varphi(X) \neq \varphi(Y)$, and then go on to show that if $\varphi(X) = \varphi(Y)$, i.e., $X \equiv Y(\Theta_{\mathbb{L}})$, then the reduction tree for F gives us a proof.

We assume that $X \neq Y$ in L. We want to show that L is indeed a countermodel for F. We do this by induction on the height n of the reduction tree P:

1. $n = 0$: The reduction tree is of the form

   $$x \leq y,$$

   but by hypothesis x and y are distinct variables, hence $F \equiv x \leq y$ is false in L.

2. $n > 0$: We distinguish cases according to the last inference in P:

   (a) $F \equiv X \cup Y \leq Z$: The last inference is $\cup$:left, P is of the form

   $$\dfrac{\begin{matrix}\vdots \\ X \leq Z\end{matrix} \quad \begin{matrix}\vdots \\ Y \leq Z\end{matrix}}{X \cup Y \leq Z}\ .$$

   Now, since every reduction branch of P is open, either the reduction tree of $X \leq Z$ or of $Y \leq Z$ must have this property as well. From the induction hypothesis we have that either $X \leq Z$ or $Y \leq Z$ is false in L. It remains to show that then $X \cup Y \leq Z$ is also false in L. W.l.o.g. assume that $X \leq Z$ is false in L.

   i. $X > Z$: Then, regardless of Y, we have $X \cup Y > Z$.

ii. X and Z are incomparable: Either $X \cup Y > Z$, which may be the case if Y and Z are compareable, or $X \cup Y$ and Z are incomparable.

(b) $F \equiv X \leq Y \cap Z$: The last inference is $\cap$:right. Similarly.

(c) $F \equiv X \cap Y \leq Z$: The last inference is a choice point for $\cap$:left, P is of the form

$$\dfrac{\begin{matrix} \vdots & & \vdots \\ X \leq Z & ; & Y \leq Z \end{matrix}}{X \cap Y \leq Z}.$$

The reduction trees of $X \leq Z$ and $Y \leq Z$ both satisfy the hypothesis (Otherwise we immediately have a closed reduction branch in P), and thus, by induction hypothesis, both $X \leq Z$ and $Y \leq Z$ are false in L, and $X \cap Y \leq Z$ is also false in L:

i. $X > Z$ and $Y > Z$: Now certainly $X \cap Y \geq Z$, and since $X \cap Y \neq Z$, we have that $X \cap Y > Z$ as well.

ii. X or Y is incomparable to Z: then $X \cap Y$ is also incomparable to Z.

We now want to prove that for formulas $F \equiv X \leq Y$ where $X \equiv Y(\Theta_{\mathbb{L}})$, the reduction tree for F contains a closed branch. We know from the completeness theorem of equational theory [1968,Theorem 26.2] that $X \equiv Y(\Theta_{\mathbb{L}})$ iff we get $X = Y$ from

$$\begin{aligned} \Sigma \quad = \quad & \{x \cup y = y \cup x, x \cap y = y \cap x, \\ & x \cup (y \cup z) = (x \cup y) \cup z, x \cap (y \cap z) = (x \cap y) \cap z, \\ & x = x \cup x, x = x \cap x, \\ & x = x \cup (x \cap y), x = x \cap (x \cup y) \\ & x = x\} \end{aligned}$$

by a applying the following rules a finite number of times:

(a) if $X = Y$, then $Y = X$

(b) if $X = Y$ and $Y = Z$, then $X = Z$

(c) if $X = Y$ and $X' = Y'$, then $X \cap X' = Y \cup Y'$ and $X \cap X' = Y \cap Y'$

(d) if $X = Y$, then $X' = Y'$, where X' and Y' are obtained from X and Y, respectively, by replacing all occurences of some variable z by an arbitrary term Z.

Obviously, we can make do without (a) if we replace Σ by Σ' obtained by adding the symmetric identities. We prove by induction on the number n of rules (b)–(d) applied that if $X = Y$, then the reduction tree for $X \leq Y$ is closed.

1. $n = 0$: the reduction trees for all the equalities in Σ' contain closed reduction branches; we list two of them:

$$\dfrac{x \leq x \quad \dfrac{x \leq x}{x \leq x \cup y}}{x \leq x \cap (x \cup y)} \qquad \dfrac{\dfrac{\dfrac{x \leq x}{x \cap y \leq x}}{(x \cap y) \cap z \leq x} \quad \dfrac{\dfrac{\dfrac{x \leq y}{x \cap y \leq y}}{(x \cap y) \cap z \leq y} \quad \dfrac{z \leq z}{(x \cap y) \cap z \leq z}}{(x \cap y) \cap z \leq y \cap z}}{(x \cap y) \cap z \leq x \cap (y \cap z)}$$

2. $n > 0$: again, we distinguish cases according to the last rule applied:

(b) The last rule applied was transitivity, $X = Z$ from $X = Y$ and $Y = Z$. The induction hypothesis applies to the two latter formulas, so we have a closed reduction branch for $X \leq Y$. In this proof tree, we follow the term Y upwards until the point where it is constructed by an application of $\substack{\cup \\ \cap}$:right:

$$\dfrac{\begin{array}{cc} & \dfrac{\vdots \quad \vdots}{X' \leq Y} \\ \vdots & \vdots \end{array}}{X \leq Y}$$

where X' is some subterm of X. We replace every occurence of Y by Z. Now either $X' < Z$, in which case we can apply the argument of the first part of the proof and obtain a closed reduction tree of $X' \leq Z$, or $X' = Z$.

(c) We have $X \cap X' = Y \cap Y'$ from $X = Y$ and $X' = Y'$, and there are closed reduction trees for $X \leq Y$ and $X' \leq Y'$. A closed reduction branch for $X \cap X' \leq Y \cap Y'$ is of the form:

$$\dfrac{\dfrac{\begin{array}{c}\vdots \\ X \leq Y\end{array}}{X \cap X' \leq Y} \quad \dfrac{\begin{array}{c}\vdots \\ X' \leq Y'\end{array}}{X \cap X' \leq Y'}}{X \cap X' \leq Y \cap Y'}$$

Similarly for $X \cup X' = Y \cup Y'$.

(d) We obtained $X' = Y'$ from $X = Y$ by replacing every occurence of some variable z by a term Z. By doing the same in the proof tree of $X \leq Y$ we obtain a proof from $Z \leq Z$. One sees by induction on the complexity k of Z, that the reduction tree for such formulas is closed: If $k = 0$, Z is a variable, and $Z \leq Z$ an axiom. If $k > 0$ and $Z \equiv X \cap Y$ the reduction tree is of the form:

$$\dfrac{\dfrac{\begin{array}{c}\vdots \\ X \leq X\end{array}}{X \cap Y \leq X} \quad \dfrac{\begin{array}{c}\vdots \\ Y \leq Y\end{array}}{X \cap Y \leq Y}}{X \cap Y \leq X \cap Y}$$

X and Y are of complexity $k - 1$, so the induction hypothesis applies. Similarly for $Z \equiv X \cup Y$.

Since the proof tree obtained by the above procedure is finite, and the number of all such trees is also finite, we see that it also is a decision method for this type of inequalities: Just construct all possible trees. If one of them is a proof tree, the formula is true, otherwise it is false.

It might be interesting to point out that this procedure gives us all counterexamples for a given false formula, and that it also shows precisely on which variables the truth or falsity of F depends (namely those occuring in the non-axiom leaves).

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