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EPSILON THEOREMS IN INTERMEDIATE LOGICS

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Dedicated to the memory of Grisha Mints

Abstract. Any intermediate propositional logic (i.e., a logic including intuitionistic logic and contained in classical logic) can be extended to a calculus with epsilon- and tau-operators and critical formulas. For classical logic, this results in Hilbert's ε -calculus. The first and second ε -theorems for classical logic establish conservativity of the ε -calculus over its classical base logic. It is well known that the second ε -theorem fails for the intuitionistic ε -calculus, as prenexation is impossible. The paper investigates the effect of adding critical ε - and τ -formulas and using the translation of quantifiers into ε - and τ -terms to intermediate logics. It is shown that conservativity over the propositional base logic also holds for such intermediate $\varepsilon\tau$ -calculi. The “extended” first ε -theorem holds if the base logic is finite-valued Gödel-Dummett logic, fails otherwise, but holds for certain provable formulas in infinite-valued Gödel logic. The second ε -theorem also holds for finite-valued first-order Gödel logics. The methods used to prove the extended first ε -theorem for infinite-valued Gödel logic suggest applications to theories of arithmetic.

§1. Introduction. The ε -calculus was originally introduced by Hilbert as a formalization of classical first-order logic. It is a way to reduce proofs in first-order logic to proofs in propositional logic from so-called critical formulas, where the role of quantifiers is taken over by certain terms. The ε -calculus was the basis for Hilbert's approach to proof theory (in particular, consistency proofs). It still is a useful logical formalism with interesting properties and theoretical and practical applications.

The ε -calculus is formulated by allowing for terms of the form $\varepsilon_x A(x)$ for any formula $A(x)$ with x free. A formula of the form $A(t) \rightarrow A(\varepsilon_x A(x))$ is called a *critical formula* belonging to $\varepsilon_x A(x)$. A proof in the ε -calculus is a proof in the quantifier-free fragment of classical logic from critical formulas. It is now possible to define the existential quantifier by $\exists x A(x) \equiv A(\varepsilon_x A(x))$, and—in classical logic—the universal quantifier by $\forall x A(x) \equiv A(\varepsilon_x \neg A(x))$. A formula A with quantifiers can thus be translated into a formula A^ε with ε -terms but without quantifiers. A formula is provable in classical first-order logic **QC** iff its translation is provable in the ε -calculus.

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Hilbert proved two fundamental results about the ε -calculus:

1. The extended first ε -theorem: If $A(\vec{e})$ is derivable in the ε -calculus (\vec{e} a tuple of ε -terms), then there are tuples of terms $\vec{t}_1, \dots, \vec{t}_n$ such that $A(\vec{t}_1) \vee \dots \vee A(\vec{t}_n)$ is provable in propositional logic.
2. The second ε -theorem: If A^ε is the standard ε -translation of a first-order formula derivable in the ε -calculus, A is derivable in first-order logic.

The extended first ε -theorem has two important consequences. The first consequence is what Hilbert simply called the first ε -theorem: If A is ε -free and derivable in the ε -calculus, it is derivable in the quantifier-free fragment of first-order logic, the so-called elementary calculus of free variables (i.e., without critical formulas, indeed, without any use of ε -terms). This implies that the ε -calculus is conservative over propositional logic for quantifier-free formulas without identity. The second consequence is Herbrand's theorem for existential formulas: If $\exists \vec{x} A(\vec{x})$ is provable in the ε -calculus, then some disjunction $A(\vec{t}_1) \vee \dots \vee A(\vec{t}_n)$ is provable in propositional logic alone, i.e., is a tautology.¹

In contrast to other proof theoretic methods which also yield the existence of Herbrand disjunctions (such as cut-elimination), the proof based on the extended first ε -theorem, and consequently the length of the Herbrand disjunction the proof yields, is insensitive to the propositional complexity of the original proof. This is an important advantage of methods based on the ε -calculus.

Hilbert's results and proofs make essential use of classical principles, especially the law of excluded middle. The question naturally arises whether the results can also be obtained for weaker logics, and whether the same proof methods can be used, i.e., whether the use of excluded middle can be avoided. In this paper, we investigate ε -calculi for intermediate logics, i.e., logics between intuitionistic and classical logic, and specifically the question of when the extended first ε -theorem holds in such logics. Well-known examples of intermediate logics are Jankov's logic of weak excluded middle and finite- and infinite-valued Gödel-Dummett logics.

We consider only intermediate logics for two reasons. One is that Hilbert's methods rely essentially on the deduction theorem, and this holds in intermediate logics but not in many other logics. The other is that ε -calculi for intuitionistic and intermediate logics are of independent interest. What is the effect of adding ε -operators to intermediate logics? When does the extended first ε -theorem hold? When is the ε -calculus for a logic conservative over the propositional base logic? We also, for the most part, discuss only pure logics, i.e., logics without identity.

In intermediate logics it is necessary to introduce a separate τ -operator which defines the universal quantifier. Whereas $A(\varepsilon_x A(x))$ translates $\exists x A(x)$, $A(\tau_x A(x))$ translates $\forall x A(x)$. The corresponding critical formulas are those of the form $A(\tau_x A(x)) \rightarrow A(t)$. Weakening the logic makes the addition of τ necessary, since

¹See Hilbert and Bernays [1939] for the first presentation of the ε -theorems, Avigad and Zach [2020] for a survey, and Moser and Zach [2006] for a modern presentation.

the equivalence of $\forall x A(x)$ and $A(\varepsilon_x \neg A(x))$ relies on the schema of contraposition; the equivalence of $\forall x A(x)$ and $A(\tau_x A(x))$ does not. The system resulting from a propositional intermediate logic \mathbf{L} by adding ε - and τ -terms and critical formulas is called the $\varepsilon\tau$ -calculus for \mathbf{L} .

It is well known that adding the ε -operator to intuitionistic logic in a straightforward way results in translations A^ε of intuitionistically invalid formulas A becoming provable. Mints [1977], [1990] has investigated different systems based on intuitionistic logic with ε -operators which are conservative and in which the ε -theorem holds. He allows the use of ε -terms only when $\exists x A(x)$ has been established; other approaches (e.g., Shirai [1971]) use existence predicates to accomplish the same. We investigate the basic $\varepsilon\tau$ -calculus without this assumption. In our $\varepsilon\tau$ -calculi, $\varepsilon\tau$ -terms are treated syntactically as something like Skolem functions rather than semantically as choice operators.

We begin (Section 2) by introducing intermediate logics. In Section 3 we introduce the $\varepsilon\tau$ -calculus and consider the differences in formulas provable in a logic vs. those provable in the corresponding $\varepsilon\tau$ -calculus. This is essentially the question of whether the addition of ε - and τ -terms and critical formulas allows the derivation (or requires the validity) of formulas not provable in the base logic.

We investigate in detail *which* intuitionistically invalid formulas are provable in $\varepsilon\tau$ -calculi for intermediate logics in Section 4. We show that quantifier shift schemas play a special role here. All but three of these are valid in intuitionistic logic. The addition of $\varepsilon\tau$ -terms and critical formulas results in the provability of the remaining three. Consequently, no intermediate first-order logic in which one of these three quantifier shifts is unprovable can have the second ε -theorem (Proposition 4.1). This includes intuitionistic logic itself, logics complete for non-constant domain Kripke frames, and infinite-valued Gödel-Dummett logic.

In Section 5, we show that conservativity for the propositional fragment nevertheless holds for all intermediate logics (Theorem 5.3). This in itself is a surprising result, even though the proof is very easy. It holds for theorems of the logics with or without identity, however not in general for provability from theories.

In Sections 6–9 we give a complete characterization of the intermediate logics where the extended first ε -theorem holds. We show (Theorem 6.2) that whenever it holds, the underlying logic must prove (or validate) a sentence B_m of the form

$$(A_1 \rightarrow A_2) \vee (A_2 \rightarrow A_3) \vee \cdots \vee (A_m \rightarrow A_{m+1}).$$

No B_m is provable in intuitionistic logic, any logic complete for Kripke frames with branching worlds, or in infinite-valued Gödel-Dummett logic. Consequently, the extended first ε -theorem does not hold for these logics.

Provability of B_m is also a sufficient condition: We show that the extended first ε -theorem holds whenever the underlying logic proves at least one B_m . The argument follows the idea of Hilbert's proof, but does not make use of excluded middle. In order to establish the result, we provide a more fine-grained analysis of the proof of the extended first ε -theorem. We first introduce the notion of

an elimination set in Section 7: a set of terms which can replace an ε -term in a proof and render the corresponding critical formulas redundant, using only the resources of the underlying propositional logic. This is the part of Hilbert's proof that uses excluded middle. We isolate here how excluded middle is used. In Section 8, we show that logics that prove some B_m also have elimination sets.

If such elimination sets exist, the procedure given by Hilbert and Bernays can be used for a proof of the extended first ε -theorem (Section 9). This establishes the second half of the characterization, that the extended first $\varepsilon\tau$ -theorem holds in logics that prove B_m , i.e., the finite-valued Gödel logics (Theorem 9.9). By putting emphasis on elimination sets, we can also show that in logics in which the extended first ε -theorem does not hold in general, it may still hold for formulas of a special form. This allows us to show that the extended first $\varepsilon\tau$ -theorem holds for negated formulas in Jankov's logic of weak excluded middle (Theorem 9.8). We also show that the first $\varepsilon\tau$ -theorem (for theorems not containing $\varepsilon\tau$ -terms) holds for infinite-valued Gödel-Dummett logic (Theorem 9.10).

The extended first $\varepsilon\tau$ -theorem is closely related to Herbrand's theorem. We discuss this connection, as well as the second $\varepsilon\tau$ -theorem, in Section 10. We show that the second $\varepsilon\tau$ -theorem holds for any intermediate predicate logic that proves all quantifier shifts and has the extended first $\varepsilon\tau$ -theorem (Proposition 10.2), e.g., $\mathbf{QLC}_m + CD$, first-order finite-valued Gödel logic.

In the case of some proofs, it is possible to eliminate $\varepsilon\tau$ -terms in a simplified way where the cases that require the presence of B_m do not arise, and the linearity schema $Lin((A \rightarrow B) \vee (B \rightarrow A))$ is enough. Although we cannot give an independent characterization of the proofs or theorems for which this is the case, the simplified procedure will sometimes terminate and produce a Herbrand disjunction. Conversely, if a Herbrand disjunction exists, there is always a proof of the original formula for which the procedure terminates and produces the Herbrand disjunction (Section 11).

This result sheds light on the conditions under which (a version of) Hilbert's method which uses principles weaker than excluded middle produces a Herbrand disjunction. In fact, a similar method can be used to give a partial ε -elimination procedure for number theory, where linearity of the natural order of \mathbb{N} plays a similar role as the schema of linearity does in the case of logic (Section 12).

§2. Preliminaries. Intermediate propositional logics have been investigated extensively since the 1950s. Intermediate predicate logics are comparatively less well understood; however, they constitute an active area of research (see Gabbay, Shehtman, and Skvortsov [2009]). We begin by collecting some preliminary definitions.

All the logics we consider are formulated in the standard language of intuitionistic and classical logic with propositional connectives \wedge , \vee , \rightarrow , the constant \perp for absurdity, and the quantifiers \forall and \exists . $\neg A$ is defined as $A \rightarrow \perp$ and \top as $\neg \perp$. Terms and atomic formulas are defined as usual. We allow 0-place predicate symbols, i.e., propositional variables. A formula containing no quantifiers is called

Propositional logics / Axioms	
H	Intuitionistic logic
KC	Logic of weak excluded middle: $\mathbf{H} + J = \neg A \vee \neg\neg A$
LC	infinite-valued Gödel logic, linear Kripke frames $\mathbf{H} + Lin = (A \rightarrow B) \vee (B \rightarrow A)$
LC_m	m -valued Gödel logic, linear Kripke frames of length $< m$ $\mathbf{H} + B_m = (A_1 \rightarrow A_2) \vee \dots \vee (A_m \rightarrow A_{m+1})$
C	Classical logic: $\mathbf{H} + A \vee \neg A, \mathbf{H} + B_2$
First-order logics / Axioms	
QH	Intuitionistic logic
QKC	Weak excluded middle: $\mathbf{QH} + J$
QLC	Linear Kripke frames: $\mathbf{QH} + Lin$
QLC_m	$\mathbf{QH} + B_m$
G_ℝ	Gödel logic on $[0, 1]$, constant-domain linear Kripke frames $\mathbf{QLC} + CD = \forall x(A(x) \vee B) \rightarrow (\forall x A(x) \vee B)$
G₀	Gödel logic on $\{0\} \cup [1/2, 1]$ $\mathbf{QLC} + CD + K = \forall x\neg\neg A(x) \rightarrow \neg\neg\forall x A(x)$
G_m	m -valued Gödel logic: $\mathbf{QH} + B_m + CD$
QC	Classical logic

TABLE 1. Intermediate logics considered

Result		Base logic	
Conservativity over propositional logic	Yes	Any	5.3
First $\varepsilon\tau$ -Theorem	Yes	LC	9.10
	Yes	Any (for negated formulas)	9.11
Extended First ε -Theorem	No	Any except LC_m	6.6
	Yes	C	9.6
	Yes	LC, KC (for negated formulas)	9.8
	Yes	LC_m	9.9
Second $\varepsilon\tau$ -Theorem	No	G_ℝ, QLC_m	10.3
	Yes	G_m, QC	10.4

TABLE 2. Epsilon theorems for intermediate logics

quantifier-free and a quantifier-free formula with only 0-place predicate symbols is called *propositional*. For the most part we will consider pure logics, i.e., logics not involving the identity predicate $=$. We assume that function symbols are available, however.

We use A, B, \dots , as metavariables for formulas, and t, s, \dots , as metavariables for terms. We write $A(x)$ to indicate that x occurs free in A . The result of substituting s for all (free) occurrences of x in a term t or a formula A is indicated

by $t[s/x]$ or $A[s/x]$. When it is clear which variable x is intended, we write $A(s)$ for $A[s/x]$. The result of replacing every occurrence of a term t by a term s in a formula A is indicated by $A[s/t]$. A *substitution instance* of a formula A is any formula resulting from A by uniformly replacing any number of atomic formulas $P(t_1, \dots, t_n)$ by formulas $B[t_1/x_1, \dots, t_n/x_n]$ (in such a way that free variables of B are not captured by quantifiers in A of course). In particular, a substitution instance of a propositional formula A is any formula resulting from A by uniformly replacing any number of 0-place predicate symbols P by formulas B .

DEFINITION 2.1. A *propositional logic* \mathbf{L} is a set of propositional formulas closed under substitution and modus ponens, i.e., if \mathbf{L} contains $A \rightarrow B$ and B it also contains A . A *predicate logic* is a set of formulas closed under substitution, modus ponens, and the quantifier rules

$$\frac{B \rightarrow A(x)}{B \rightarrow \forall y A(y)} \quad \frac{A(x) \rightarrow B}{\exists y A(y) \rightarrow B}$$

which are subject to the eigenvariable condition: x must not be free in the conclusion.

Logics can be characterized by the formulas derivable from a set of axiom schemas by modus ponens. This is of course equivalent to closing the axioms under substitution and modus ponens. For instance, intuitionistic propositional logic \mathbf{H} is obtained from the propositional axioms given in Troelstra and van Dalen [1988, 4.1]. Classical propositional logic \mathbf{C} and predicate logic \mathbf{QC} are obtained by adding $\neg A \vee A$ (or alternatively $\neg\neg A \rightarrow A$) as an axiom.

DEFINITION 2.2. An *intermediate propositional logic* \mathbf{L} is a propositional logic that contains \mathbf{H} and is contained in \mathbf{C} .

The following intermediate propositional logics will play important roles:

1. \mathbf{LC} , characterized alternatively as the formulas valid on linearly ordered Kripke frames or as infinite-valued Gödel logic,² axiomatized over \mathbf{H} using the schema

$$(A \rightarrow B) \vee (B \rightarrow A). \quad (Lin)$$

2. $\mathbf{LC}_m = \mathbf{LC} + B_m$, characterized as formulas valid on linearly ordered Kripke frames of height $< m$, or as the Gödel logic on m truth values, also known as \mathbf{S}_{m-1} [Hosoi 1966]. Here, B_m is:

$$(A_1 \rightarrow A_2) \vee (A_2 \rightarrow A_3) \vee \dots \vee (A_m \rightarrow A_{m+1}). \quad (B_m)$$

3. \mathbf{KC} , the logic of weak excluded middle [Jankov 1968], axiomatized over \mathbf{H} using the schema

$$\neg A \vee \neg\neg A. \quad (J)$$

²See [Dummett 1959]. Note that although there is only one infinite-valued Gödel logic considered as a set of tautologies, there are infinitely many different consequence relations on infinite truth-value sets with the Gödel truth functions [Baaz and Zach 1998].

If \mathbf{L} is an intermediate propositional logic, we can consider the corresponding “elementary calculus,” i.e., the system obtained by replacing propositional variables with atomic formulas of a first-order language, with or without identity. We will be interested in formulas provable in such an elementary calculus from a set of assumptions Γ .

DEFINITION 2.3. Suppose \mathbf{L} is an intermediate logic. A proof π of A from Γ in the corresponding *elementary calculus* is a sequence $A_1, \dots, A_n = A$ of quantifier-free formulas such that each A_i is either a substitution instance of a formula in \mathbf{L} , is in Γ , or follows from formulas A_k and A_l ($k, l < i$) by modus ponens, and $A_n \equiv A$. We then write $\Gamma \vdash_{\mathbf{L}}^{\pi} A$. If such a proof π exists we write $\Gamma \vdash_{\mathbf{L}} A$ and if Γ is empty, simply $\mathbf{L} \vdash A$. If \mathbf{L} is not mentioned, we mean \mathbf{H} .

Given a propositional logic \mathbf{L} , and possibly a set of additional axiom schemas Ax involving quantifiers, we can generate a predicate logic:

DEFINITION 2.4. If \mathbf{L} is a propositional logic, and Ax is a set of formulas, then $\mathbf{QL} + Ax$ is the smallest predicate logic containing \mathbf{L} , the standard quantifier axioms

$$\begin{aligned} \forall x A(x) \rightarrow A(t) \quad \text{and} \\ A(t) \rightarrow \exists x A(x), \end{aligned}$$

and the formulas in Ax .

If $\mathbf{QL} + Ax$ contains intuitionistic predicate logic \mathbf{QH} and is contained in classical predicate logic \mathbf{QC} , it is called an *intermediate predicate logic*.

DEFINITION 2.5. Suppose \mathbf{L} is a propositional logic. A proof π of A in $\mathbf{QL} + Ax$ is a sequence $A_1, \dots, A_n = A$ of formulas of predicate logic such that each A_i is either a substitution instance of a formula in \mathbf{L} , of a standard quantifier axiom, of a formula in Ax , or follows from previous formulas by modus ponens or a quantifier rule. We then write $\mathbf{QL} + Ax \vdash^{\pi} A$. If such a proof π exists we write $\mathbf{QL} + Ax \vdash A$.

A is a formula in $\mathbf{QL} + Ax$ iff $\mathbf{QL} + Ax \vdash A$. If \mathbf{L} itself is characterized by a set of propositional axioms, it is enough to require substitution instances of axioms of \mathbf{L} in the above definition. For instance, taking \mathbf{H} as above, the proofs in intuitionistic predicate logic \mathbf{QH} are just the proofs in the system $H_2\text{-IQC}$ of Troelstra and van Dalen [1988, 4.3].

If Ax is empty, $\mathbf{QL} = \mathbf{QL} + Ax$ is the weakest pure intermediate predicate logic extending \mathbf{L} . For instance, \mathbf{QLC} is the weakest predicate logic obtained from \mathbf{LC} , and is axiomatized by $\mathbf{QH} + Lin$. It is complete for linearly-ordered Kripke frames (see Corsi [1992], Skvortsov [2005]).

It is possible to extend the weakest predicate logic of an intermediate propositional logic \mathbf{L} by adding intuitionistically invalid schemas Ax involving quantifiers. Some important examples are the constant domain principle,

$$\forall x(A(x) \vee B) \rightarrow (\forall x A(x) \vee B), \quad (CD)$$

the double negation shift (or Kuroda's principle),

$$\forall x \neg\neg A(x) \rightarrow \neg\neg\forall x A(x) \quad (K)$$

and the quantifier shifts

$$(B \rightarrow \exists x A(x)) \rightarrow \exists x(B \rightarrow A(x)) \quad (Q\exists)$$

$$(\forall x A(x) \rightarrow B) \rightarrow \exists x(B \rightarrow A(x)) \quad (Q\forall)$$

QLC + *CD* axiomatizes the formulas valid in linearly-ordered Kripke frames with constant domains, and also the first-order Gödel logic $\mathbf{G}_{\mathbb{R}}$ of formulas valid on the interval $[0, 1]$. **QH** + *K* characterizes the formulas valid on Kripke frames with the McKinsey property and **QLC** + *CD* + *K* is the logic of linear Kripke frames with maximal element. It is also the first-order Gödel logic \mathbf{G}_0 of formulas valid on $\{0\} \cup [1/2, 1]$. **QLC**_{*m*} is not complete for linear Kripke frames of height $< m$ (contrary to what one might expect). However, $\mathbf{G}_m = \mathbf{QLC}_m + CD$ is complete for linearly ordered Kripke frames of height $< m$ with constant domains. It is also the *m*-valued first-order Gödel logic \mathbf{G}_m .³

§3. $\varepsilon\tau$ -Calculi for Intermediate Logics. Formulas and terms of the $\varepsilon\tau$ -calculus are defined by simultaneous induction, allowing that if $A(x)$ is a formula already defined, then $\varepsilon_x A(x)$ and $\tau_x A(x)$ are terms. In $\varepsilon_x A(x)$ and $\tau_x A(x)$ the variable x is bound. We call terms of the form $\varepsilon_x A(x)$ *ε -terms* and those of the form $\tau_x A(x)$, *τ -terms* (collectively: *$\varepsilon\tau$ -terms*).

As usual, we consider ε - and τ -terms to be identical up to renaming of bound variables, and define substitution of $\varepsilon\tau$ -terms into formulas, as in $A(\varepsilon_x A(x))$, so that bound variables are tacitly renamed to avoid clashes.

DEFINITION 3.1. A *critical formula belonging to $\varepsilon_x A(x)$* is any formula of the form $A(t) \rightarrow A(\varepsilon_x A(x))$.

A critical formula belonging to $\tau_x A(x)$ is any formula of the form $A(\tau_x A(x)) \rightarrow A(t)$.

DEFINITION 3.2. Suppose \mathbf{L} is an intermediate propositional logic. An *$\varepsilon\tau$ -proof π* of B is a proof in the elementary calculus of \mathbf{L} from critical formulas Γ of the form

$$\begin{aligned} A(t) &\rightarrow A(\varepsilon_x A(x)) \\ A(\tau_x A(x)) &\rightarrow A(t). \end{aligned}$$

We write $\mathbf{L}\varepsilon\tau \vdash B$ if such a π exists, or $\Gamma \vdash_{\mathbf{L}\varepsilon\tau}^{\pi} B$ when we want to identify the critical formulas and the proof π .

The *pure $\varepsilon\tau$ -calculus* $\mathbf{L}\varepsilon\tau$ of \mathbf{L} is the set of quantifier-free formulas that have $\varepsilon\tau$ -proofs.

³See Skvortsov [2005] and Gabbay, Shehtman, and Skvortsov [2009] for the mentioned Kripke completeness results and Baaz, Preining, and Zach [2007] for the characterizations in terms of Gödel truth value sets. (\mathbf{G}_m , $\mathbf{G}_{\mathbb{R}}$, and \mathbf{G}_0 are the only axiomatizable first-order Gödel logics.)

We are interested in the relationships between intermediate predicate logics $\mathbf{QL} + Ax$ and the $\varepsilon\tau$ -calculus $\mathbf{L}\varepsilon\tau$ of their propositional fragment \mathbf{L} . Since the language of $\mathbf{L}\varepsilon\tau$ does not contain quantifiers, we must define a translation of predicate formulas that do contain them into the language of the $\varepsilon\tau$ -calculus.

DEFINITION 3.3. The $\varepsilon\tau$ -translation $A^{\varepsilon\tau}$ of a formula A is defined as follows:

$$\begin{aligned} A^{\varepsilon\tau} &= A \text{ if } A \text{ is atomic} \\ (A \wedge B)^{\varepsilon\tau} &= A^{\varepsilon\tau} \wedge B^{\varepsilon\tau} & (A \vee B)^{\varepsilon\tau} &= A^{\varepsilon\tau} \vee B^{\varepsilon\tau} \\ (A \rightarrow B)^{\varepsilon\tau} &= A^{\varepsilon\tau} \rightarrow B^{\varepsilon\tau} & (\neg A)^{\varepsilon\tau} &= \neg A^{\varepsilon\tau} \\ (\exists x A(x))^{\varepsilon\tau} &= A^{\varepsilon\tau}(\varepsilon_x A(x)^{\varepsilon\tau}) & (\forall x A(x))^{\varepsilon\tau} &= A^{\varepsilon\tau}(\tau_x A(x)^{\varepsilon\tau}) \end{aligned}$$

Again, substitution of $\varepsilon\tau$ -terms for variables must be understood modulo re-naming of bound variables so as to avoid clashes. Clearly, if A contains no quantifiers, then $A^{\varepsilon\tau} = A$.

The point of the classical ε -calculus is that it can *replace* quantifiers and quantifier inferences. And indeed, in classical first-order logic, a first-order formula A is provable iff its translation A^ε is provable in the pure ε -calculus. The “if” direction is the content of the second ε -theorem, while the “only if” direction follows more simply by translating derivations.

We defined the $\varepsilon\tau$ -calculus on the basis of a propositional logic \mathbf{L} . It is also possible to define an “extended” $\varepsilon\tau$ -calculus by adding $\varepsilon\tau$ -terms and critical formulas to the full first-order language including quantifiers, and then considering proofs in $\mathbf{QL} + Ax$ from critical formulas.

DEFINITION 3.4. A proof π in $(\mathbf{QL} + Ax)\varepsilon\tau$ of A is a sequence of formulas of the $\varepsilon\tau$ -calculus (possibly containing quantifiers) ending in A in which each formula is either an instance of a formula in \mathbf{L} , an instance of a schema in Ax , a standard quantifier axiom, a critical formula, or follows from preceding formulas by modus ponens or a quantifier rule. We write $(\mathbf{QL} + Ax)\varepsilon\tau \vdash A$ if such a π exists.

The *extended $\varepsilon\tau$ -calculus* $(\mathbf{QL} + Ax)\varepsilon\tau$ of $\mathbf{QL} + Ax$ is the set of formulas that have proofs in $(\mathbf{QL} + Ax)\varepsilon\tau$.

If A is quantifier-free and $\mathbf{L}\varepsilon\tau \vdash A$ then $(\mathbf{QL} + Ax)\varepsilon\tau \vdash A$. One may wonder, however, if $(\mathbf{QL} + Ax)\varepsilon\tau$ is stronger than $\mathbf{L}\varepsilon\tau$ in the sense that for some formulas A , $(\mathbf{QL} + Ax)\varepsilon\tau \vdash A$ but not $\mathbf{L}\varepsilon\tau \vdash A^{\varepsilon\tau}$. This is not so as long as the $\varepsilon\tau$ -translations of the axioms Ax of $\mathbf{QL} + Ax$ are provable in $\mathbf{L}\varepsilon\tau$; then the extended ε -calculus is conservative over the pure ε -calculus.

LEMMA 3.5. *Suppose $\mathbf{QL} + Ax$ is an intermediate predicate logic, and for each quantifier axiom $B \in Ax$, $\mathbf{L}\varepsilon\tau \vdash B^{\varepsilon\tau}$. If $(\mathbf{QL} + Ax)\varepsilon\tau \vdash A$, then $\mathbf{L}\varepsilon\tau \vdash A^{\varepsilon\tau}$.*

PROOF. By standard proof transformations we may assume that the proof π in $(\mathbf{QL} + Ax)\varepsilon\tau$ is such that every formula is used as a premise of at most one modus ponens or quantifier inference, and that the eigenvariables of all quantifier inferences are distinct (the proof is *regular*). The proof then proceeds by induction on the length of π .

Any formula B in π that is not the conclusion of an inference is either in \mathbf{L} , a critical formula, in Ax , or a standard quantifier axiom. Then $B^{\varepsilon\tau}$ is also either an axiom in \mathbf{L} , a critical formula (as can easily be seen from the definition of the $\varepsilon\tau$ -translation), or, if $B \in Ax$ then $\mathbf{L}\varepsilon\tau \vdash B^{\varepsilon\tau}$ (by hypothesis). If B is a standard quantifier axiom, its $\varepsilon\tau$ -translation is a critical formula:

$$\begin{aligned} [A(t) \rightarrow \exists x A(x)]^{\varepsilon\tau} &= A^{\varepsilon\tau}(t) \rightarrow A^{\varepsilon\tau}(\varepsilon_x A^{\varepsilon\tau}(x)) \\ [\forall x A(x) \rightarrow A(t)]^{\varepsilon\tau} &= A^{\varepsilon\tau}(\tau_x A^{\varepsilon\tau}(x)) \rightarrow A^{\varepsilon\tau}(t) \end{aligned}$$

If $B \rightarrow A(x)$ is derivable (where the eigenvariable x is not free in B), then so is $B \rightarrow A(\tau_x A(x))$, by substituting $\tau_x A(x)$ everywhere x appears free in the part of π leading to $B \rightarrow A(x)$, and renaming bound variables to avoid clashes. Thus, if B is the conclusion of a quantifier inference, $\mathbf{L}\varepsilon\tau \vdash B^{\varepsilon\tau}$. Similarly, if $B(x) \rightarrow A$ is derivable, so is $B(\varepsilon_x B(x)) \rightarrow A$. (Cf. Moser and Zach [2006, Lemma 7].) \dashv

As we'll see in Section 4, the quantifier axioms of the intermediate predicate logics considered in the preceding section satisfy the condition that the $\varepsilon\tau$ -translations of their additional quantifier axioms are derivable from critical formulas alone (i.e., already in $\mathbf{L}\varepsilon\tau$).

COROLLARY 3.6. *If $\mathbf{QL} + Ax$ satisfies the conditions of Lemma 3.5, and A contains no quantifiers, then $(\mathbf{QL} + Ax)\varepsilon\tau \vdash A$ only if $\mathbf{L}\varepsilon\tau \vdash A$.*

PROOF. If A contains no quantifiers, then $A^{\varepsilon\tau} \equiv A$. \dashv

PROPOSITION 3.7. *For any intermediate predicate logic $\mathbf{QL} + Ax$,*

$$\begin{aligned} (\mathbf{QL} + Ax)\varepsilon\tau \vdash \forall x A(x) &\leftrightarrow A(\tau_x A(x)) \text{ and} \\ (\mathbf{QL} + Ax)\varepsilon\tau \vdash \exists x A(x) &\leftrightarrow A(\varepsilon_x A(x)). \end{aligned}$$

PROOF. In each case, one direction is an instance of the corresponding quantifier axiom, and the other direction follows from a critical formula by the corresponding quantifier rule. For instance, $A(\varepsilon_x A(x)) \rightarrow \exists x A(x)$ is a standard quantifier axiom, and from the critical formula $A(x) \rightarrow A(\varepsilon_x A(x))$ we get $\exists x A(x) \rightarrow A(\varepsilon_x A(x))$ by the \exists -rule, since x is not free in $A(\varepsilon_x A(x))$. \dashv

PROPOSITION 3.8. *If $\mathbf{QL} + Ax$ satisfies the conditions of Lemma 3.5, then $(\mathbf{QL} + Ax)\varepsilon\tau \vdash A \leftrightarrow A^{\varepsilon\tau}$.*

PROOF. Since $\mathbf{QL} + Ax$ includes \mathbf{QH} , the substitution rule $B \leftrightarrow C \vdash D(B) \leftrightarrow D(C)$ is admissible. The result follows by induction on complexity of A and the previous proposition. \dashv

§4. Critical Formulas and Quantifier Shifts. We will show later (Theorem 5.3) that any $\varepsilon\tau$ -calculus for an intermediate logic \mathbf{L} is conservative over \mathbf{L} . It is well-known that the $\varepsilon\tau$ -calculus over intuitionistic logic is not conservative over intuitionistic *predicate* logic. We'll show now specifically that for any intermediate logic \mathbf{L} , the $\varepsilon\tau$ -translations of all classically valid quantifier shift principles are provable from critical formulas.

	$C(x)$	t_1	t_2	
$\forall x(A(x) \vee B) \rightarrow (\forall x A(x) \vee B)$	$A(x) \vee B$	$\tau_x C(x)$	$\tau_x A(x)$	(CD)
$(\exists x A(x) \vee B) \rightarrow \exists x(A(x) \vee B)$	$A(x) \vee B$	$\varepsilon_x A(x)$	$\varepsilon_x C(x)$	
$\forall x(A(x) \wedge B) \rightarrow (\forall x A(x) \wedge B)$	$A(x) \wedge B$	$\tau_x C(x)$	$\tau_x A(x)$	
$(\exists x A(x) \wedge B) \rightarrow \exists x(A(x) \wedge B)$	$A(x) \wedge B$	$\varepsilon_x A(x)$	$\varepsilon_x C(x)$	
$(B \rightarrow \exists x A(x)) \rightarrow \exists x(B \rightarrow A(x))$	$B \rightarrow A(x)$	$\varepsilon_x A(x)$	$\varepsilon_x C(x)$	(Q_{\exists})
$\forall x(B \rightarrow A(x)) \rightarrow (B \rightarrow \forall x A(x))$	$B \rightarrow A(x)$	$\tau_x C(x)$	$\tau_x A(x)$	
$(\forall x A(x) \rightarrow B) \rightarrow \exists x(A(x) \rightarrow B)$	$A(x) \rightarrow B$	$\tau_x A(x)$	$\varepsilon_x C(x)$	(Q_{\forall})
$\forall x(A(x) \rightarrow B) \rightarrow (\exists x A(x) \rightarrow B)$	$A(x) \rightarrow B$	$\tau_x C(x)$	$\varepsilon_x A(x)$	
$\forall x \neg \neg A(x) \rightarrow \neg \neg \forall x A(x)$	$\neg \neg A(x)$	$\tau_x \neg \neg A(x)$	$\tau_x A(x)$	(K)

TABLE 3. Quantifier shift formulas whose $\varepsilon\tau$ -translations are critical formulas. In each case, x is not free in B , and the $\varepsilon\tau$ -translation of the quantifier shift formula on the left is $C(t_1) \rightarrow C(t_2)$.

Quantifier shift formulas divide into two kinds. On the one hand, we have conditionals the $\varepsilon\tau$ -translations of which are critical formulas, and are therefore provable in $\mathbf{L}\varepsilon\tau$ (see Table 3). On the other hand, we have formulas provable from critical formulas together with some propositional principles, all of which are intuitionistically valid and hence provable in all intermediate logics. For instance, to obtain the $\varepsilon\tau$ -translation of

$$(\forall x A(x) \vee B) \rightarrow \forall x(A(x) \vee B),$$

take $A_1 = A(\tau_x A(x))$ and $A_2 = A(\tau_x (A(x) \vee B))$. Then $A_1 \rightarrow A_2$ is a critical formula, viz.,

$$A(\tau_x A(x)) \rightarrow A(\tau_x (A(x) \vee B)).$$

Apply modus ponens to it and the principle

$$(A_1 \rightarrow A_2) \rightarrow ((A_1 \vee B) \rightarrow (A_2 \vee B)).$$

This same pattern works in all cases, the required critical formulas $A_1 \rightarrow A_2$ and propositional principles are given in Table 4.

The most interesting quantifier shift formulas here are CD , Q_{\forall} , and Q_{\exists} , since they are not intuitionistically valid. By contrast, we have:

PROPOSITION 4.1. *If \mathbf{L} is an intermediate propositional logic, then:*

1. $K^{\varepsilon\tau}$, $CD^{\varepsilon\tau}$, $(Q_{\exists})^{\varepsilon\tau}$, and $(Q_{\forall})^{\varepsilon\tau}$ are provable in $\mathbf{L}\varepsilon\tau$.
2. K , CD , Q_{\exists} , and Q_{\forall} are provable in $(\mathbf{QL} + Ax)\varepsilon\tau$.

PROOF. (1) They are critical formulas; see Table 3.

$(\forall x A(x) \vee B) \rightarrow \forall x(A(x) \vee B)$	$A_1 = A(\tau_x A(x)), A_2 = A(\tau_x (A \vee B))$
$\exists x(A(x) \vee B) \rightarrow (\exists x A(x) \vee B)$	$A_1 = A(\varepsilon_x (A(x) \vee B)), A_2 = A(\varepsilon_x A(x))$
$(A_1 \vee B) \rightarrow (A_2 \vee B)$	$(A_1 \rightarrow A_2) \rightarrow ((A_1 \vee B) \rightarrow (A_2 \vee B))$
$\forall x(A(x) \wedge B) \rightarrow (\forall x A(x) \wedge B)$	$A_1 = A(\tau_x A(x)), A_2 = A(\tau_x (A(x) \wedge B))$
$\exists x(A(x) \wedge B) \rightarrow (\exists x A(x) \wedge B)$	$A_1 = A(\varepsilon_x (A(x) \wedge B)), A_2 = A(\varepsilon_x A(x))$
$(A_1 \wedge B) \rightarrow (A_2 \wedge B)$	$(A_1 \rightarrow A_2) \rightarrow ((A_1 \wedge B) \rightarrow (A_2 \wedge B))$
$\exists x(B \rightarrow A(x)) \rightarrow (B \rightarrow \exists x A(x))$	$A_1 = A(\varepsilon_x (B \rightarrow A(x))), A_2 = A(\varepsilon_x A(x))$
$(B \rightarrow \forall x A(x)) \rightarrow \forall x(B \rightarrow A(x))$	$A_1 = A(\tau_x A(x)), A_2 = A(\tau_x (B \rightarrow A(x)))$
$(B \rightarrow A_1) \rightarrow (B \rightarrow A_2)$	$(A_1 \rightarrow A_2) \rightarrow ((B \rightarrow A_1) \rightarrow (B \rightarrow A_2))$
$\exists x(A(x) \rightarrow B) \rightarrow (\forall x A(x) \rightarrow B)$	$A_1 = A(\tau_x A(x)), A_2 = A(\varepsilon_x (A(x) \rightarrow B))$
$(\exists x A(x) \rightarrow B) \rightarrow \forall x(A(x) \rightarrow B)$	$A_1 = A(\tau_x (A(x) \rightarrow B)), A_2 = A(\varepsilon_x A(x))$
$(A_1 \rightarrow B) \rightarrow (A_2 \rightarrow B)$	$(A_1 \rightarrow A_2) \rightarrow ((A_2 \rightarrow B) \rightarrow (A_1 \rightarrow B))$

TABLE 4. Proofs of $\varepsilon\tau$ -translations of quantifier shift formulas. In each case, x is not free in B , $A_1 \rightarrow A_2$ is a critical formula, the $\varepsilon\tau$ -translation of the formula is given on the left. The propositional principle on the right is provable in intuitionistic logic, and the $\varepsilon\tau$ -translation of the quantifier shift formula follows by one application of modus ponens.

(2) Follows from Proposition 3.8. ⊣

The second ε -theorem states that if A^ε is provable in the pure ε -calculus, then A is provable in classical predicate logic. The second ε -theorem fails for any intermediate predicate logic $\mathbf{QL} + Ax$, in which $A^{\varepsilon\tau}$ is provable in $\mathbf{L}\varepsilon\tau$ but A is not provable in $\mathbf{QL} + Ax$, e.g., when $\mathbf{QL} + Ax$ does not prove one of K , CD , Q_\exists , or Q_\forall .

Note that the only intuitionistically invalid De Morgan rule for quantifiers,

$$\neg\forall x A(x) \rightarrow \exists x \neg A(x), \tag{Q}$$

is a special case of Q_\forall , taking \perp for B ; $Q^{\varepsilon\tau}$ is a critical formula. The $\varepsilon\tau$ -translation of double negation shift $K^{\varepsilon\tau}$ is

$$\neg\neg A(\tau_x \neg\neg A(x)) \rightarrow \neg\neg A(\tau_x A(x))$$

and is also a critical formula.

In classical first-order logic, both the addition of ε -operators and critical formulas and the replacement of quantifiers by ε -operators is conservative. The previous results show that for extensions of first-order intuitionistic logic, this is not the case: intuitionistically invalid quantified formulas (or their $\varepsilon\tau$ -translations) become provable. However, these quantifier shifts *are* provable in some intermediate logics, e.g., in some Gödel logics.

We might think of $\varepsilon\tau$ -terms semantically as terms for objects which serve the role of generics taking on the role of quantifiers, and indeed in classical logic this connection is very close. Because of the validity of

$$\exists x(\exists y A(y) \rightarrow A(x)) \quad (Wel_1)$$

$$\exists x(A(x) \rightarrow \forall y A(y)) \quad (Wel_2)$$

in classical logic, there always is an object x which behaves as an ε -term ($A(x)$ holds iff $\exists x A(x)$ holds), and an object x which behaves as a τ -term (i.e., $A(x)$ holds iff $\forall y A(y)$ holds). One might expect then that Wel_1 and Wel_2 , when added to **QH**, have the same effect as adding critical formulas, i.e., that all quantifier shifts become provable. Note that Wel_1 and Wel_2 are intuitionistically equivalent to

$$\exists x\forall y(A(y) \rightarrow A(x)) \quad (Wel'_1)$$

$$\exists x\forall y(A(x) \rightarrow A(y)). \quad (Wel'_2)$$

As is easily checked, **QH** $\vdash Wel_1 \leftrightarrow Q_{\exists}$ and **QH** $\vdash Wel_2 \leftrightarrow Q_{\forall}$, and even **QH** $\vdash Wel_2 \rightarrow CD$. However, **QH** $\not\vdash Wel_1 \leftrightarrow CD$.⁴

§5. $L\varepsilon\tau$ is Conservative over L . The classical ε -calculus is conservative over propositional logic. Work by Bell [1993] and DeVidi [1995] shows that, however, the addition of critical formulas to intuitionistic logic results in intuitionistically invalid *propositional* formulas becoming provable in certain simple theories. These results require the presence of identity axioms. One may wonder if these results can be strengthened to the pure logic and the $\varepsilon\tau$ -calculus alone. The following proposition shows that this is not the case. The addition of critical formulas to intermediate logics alone does not have any effects on the propositional level.

DEFINITION 5.1. The *shadow* A^s of a formula is defined as follows:

$$\begin{aligned} P(t_1, \dots, t_n)^s &= X_P \\ (t_1 = t_2)^s &= \top \\ (A \wedge B)^s &= A^s \wedge B^s & (A \vee B)^s &= A^s \vee B^s \\ (A \rightarrow B)^s &= A^s \rightarrow B^s & (\neg A)^s &= \neg A^s \\ (\exists x A(x))^s &= A(x)^s & (\forall x A(x))^s &= A(x)^s \end{aligned}$$

where X_P is a propositional variable and \top is any theorem of **L**.

The shadow of a proof $\pi = A_1, \dots, A_n$ is A_1^s, \dots, A_n^s .

A first-order intermediate logic **QL** $+ Ax$ (over a propositional base logic **L**) is *preserved under shadow* if **L** $\vdash B^s$ for all quantifier axioms $B \in Ax$.

The shadow of a formula is a propositional formula obtained by disregarding all first-order structure. If an intermediate predicate logic **QL** $+ Ax$ is preserved under shadow, the shadows of its theorems are already valid in **L**.

⁴See p. 694 of Skvortsov [2006].

PROPOSITION 5.2. *Suppose $\mathbf{QL} + Ax$ is preserved under shadow. If $A_1, \dots, A_n \vdash_{(\mathbf{QL} + Ax)\varepsilon\tau} B$, then $A_1^s, \dots, A_n^s \vdash_{\mathbf{L}} B^s$. This also holds if identity axioms are present.*

PROOF. Consider a derivation π in $(\mathbf{QL} + Ax)\varepsilon\tau$ of B from A_1, \dots, A_n , and a formula C in π not a conclusion of an inference, and not among A_1, \dots, A_n . If $C \in \mathbf{L}$, then also $C^s \in \mathbf{L}$. If C is a critical formula, then C^s is of the form $A \rightarrow A$. If C is a standard quantifier axiom, we have $(\forall x A \rightarrow A(t))^s \equiv (A(t) \rightarrow \exists x A(x))^s \equiv A(x)^s \rightarrow A(x)^s$, which again is in \mathbf{L} . (Clearly, $A(t)^s \equiv A(x)^s$.)

If C is the conclusion of modus ponens from premises A and $A \rightarrow C$, then C^s follows from A^s and $(A \rightarrow C)^s$ by modus ponens. If C is the conclusion of a quantifier rule, the shadows of premise and conclusion are identical, e.g.,

$$(B \rightarrow A(x))^s \equiv B^s \rightarrow A(x)^s \equiv (B \rightarrow \forall x A(x))^s$$

Thus we have shown that π^s is a derivation of B^s from A_1^s, \dots, A_n^s in \mathbf{L} .

This still holds if identity is present, as the shadows of identity axioms are: $(t = t)^s \equiv \top$ and

$$(t_1 = t_2 \rightarrow (A(t_1) \rightarrow A(t_2)))^s \equiv \top \rightarrow (A(t_1)^s \rightarrow A(t_1)^s)$$

since $A(t_1)^s \equiv A(t_2)^s$. Both are provable in \mathbf{H} and thus in \mathbf{L} . \dashv

All intermediate predicate logics mentioned above are preserved under shadow. They are axiomatized by various quantifier shift principles. As we have seen in the preceding section, the $\varepsilon\tau$ -translations of all such quantifier shift principles become provable in the corresponding $\varepsilon\tau$ -calculus. However, the shadow of such a quantifier shift principle is a formula of the form $B \rightarrow B$. As a consequence, we have the following conservativity result for all intermediate $\varepsilon\tau$ -calculi:

THEOREM 5.3. *If $\mathbf{QL} + Ax$ is preserved under shadow, then $(\mathbf{QL} + Ax)\varepsilon\tau$ is conservative over \mathbf{L} for propositional formulas. In particular, no new propositional formulas become provable in $\mathbf{L}\varepsilon\tau$ by the addition of critical formulas to any intermediate logic \mathbf{L} , including intuitionistic logic itself.*

Bell [1993] claimed that in the extended intuitionistic ε -calculus for \mathbf{QH} with identity, we have $D \vdash_{\mathbf{QH}\varepsilon\tau} M$, where M is

$$\neg(B \wedge C) \rightarrow (\neg B \vee \neg C) \tag{M}$$

and D is $\forall x(x = a \vee \neg x = a)$. M is an intuitionistically invalid direction of De Morgan's laws. Since the shadow D^s of D is the intuitionistically valid formula $\top \vee \neg\top$, this seems to contradict Lemma 5.2. The proof starts by asserting that

$$\forall x [(x = a \wedge B) \vee (x \neq a \wedge C)] \rightarrow (B \wedge C)$$

is provable in \mathbf{QH} with identity. This is false, however, as the formula is not true in any one-element model when B is true and C is false. Theorem 7 of DeVidi

[1995] fails for the same reason. The results are correct with the additional assumption $a \neq b$.⁵

§6. The Extended First $\varepsilon\tau$ -Theorem Fails unless $\mathbf{L} \vdash B_m$. In classical first-order logic, the main result about the ε -calculus is the extended first ε -theorem. It states that if $A(e_1, \dots, e_n)$, where the e_i are ε -terms, is provable in the pure ε -calculus, then there are ε -free terms t_i^j such that

$$A(t_i^1, \dots, t_n^1) \vee \dots \vee A(t_i^k, \dots, t_n^k)$$

is provable in classical propositional logic alone. Such ε -terms e_i appear as the result of translating $\exists x_1 \dots \exists x_n A(x_1, \dots, x_n)$ into the ε -calculus.

In the context of intermediate logics, we may formulate the statement as follows:

DEFINITION 6.1. An intermediate logic \mathbf{L} has the extended first $\varepsilon\tau$ -theorem, if, whenever $\mathbf{L}\varepsilon\tau \vdash A(e_1, \dots, e_n)$ for some ε - or τ -terms e_1, \dots, e_n , then there are $\varepsilon\tau$ -free terms t_i^j such that

$$\mathbf{L} \vdash A(t_i^1, \dots, t_n^1) \vee \dots \vee A(t_i^k, \dots, t_n^k).$$

We obtain a first *negative* result: if $\mathbf{L}\varepsilon\tau$ has the extended first epsilon theorem, then an instance of B_m , i.e.,

$$(A_1 \rightarrow A_2) \vee \dots \vee (A_m \rightarrow A_{m+1})$$

for some $m \geq 2$ is provable already in the propositional fragment \mathbf{L} .⁶ This rules out an extended first $\varepsilon\tau$ -theorem for, e.g., $\varepsilon\tau$ -calculi for intuitionistic logic and infinite-valued Gödel-Dummett logic.

THEOREM 6.2. Suppose $\mathbf{L}\varepsilon\tau$ has the extended first $\varepsilon\tau$ -theorem. Then $\mathbf{L} \vdash B_m$ for some $m \geq 2$.

PROOF. Consider Let $A(z) \equiv (P(f(z)) \rightarrow P(z))$ and $\exists z A(z)$, i.e., $\exists z (P(f(z)) \rightarrow P(z))$. Let $e \equiv \varepsilon_z (P(f(z)) \rightarrow P(z))$. The $\varepsilon\tau$ -translation of $\exists z A(z)$ is $P(f(e)) \rightarrow P(e)$, i.e.,

$$V \equiv P(f(\varepsilon_z (P(f(z)) \rightarrow P(z)))) \rightarrow P(\varepsilon_z (P(f(z)) \rightarrow P(z)))$$

Let $U \equiv A(\varepsilon_x P(x)) \equiv P(f(\varepsilon_x P(x))) \rightarrow P(\varepsilon_x P(x))$. Note that U is of the form $P(t) \rightarrow P(\varepsilon_x P(x))$, so it is a critical formula. Also note that $U \rightarrow V$ is of the form $A(t) \rightarrow A(e)$, and so $U \rightarrow V$ is also a critical formula.

⁵Bell provides another proof of M in intuitionistic ε -calculus which explicitly requires, in addition to D , the assumption $a \neq b$. DeVidi [1995] shows that in the intuitionistic $\varepsilon\tau$ -calculus, $D \wedge a \neq b$ derives Lin . (Note that also $\mathbf{LC} \vdash M$.) However, since the shadow of $a \neq b$ is $\neg\top$, these proofs do not conflict with our Lemma 5.2. Bell's other examples of intuitionistically invalid propositional formulas provable in $\varepsilon\tau$ -calculi all require assumptions of the form $a \neq b$ and also the axiom of ε -extensionality. The examples of derivations of M and Lin in intuitionistic $\varepsilon\tau$ -calculus given by Mulvihill [2015] avoid identity but require the assumptions $\forall x((P(x) \rightarrow P(a)) \vee \neg(P(x) \rightarrow P(a)))$ and $\neg(P(a) \rightarrow P(b))$.

⁶For $m = 2$, this schema is equivalent to $A \vee \neg A$: take \top for A_1 , A for A_2 , \perp for A_3 .

Since $\mathbf{L} \vdash (U \rightarrow V) \rightarrow (U \rightarrow V)$, and $U \rightarrow V$ and U are critical formulas, $\mathbf{L}\varepsilon\tau \vdash V$. By assumption, $\mathbf{L}\varepsilon\tau$ has the extended first $\varepsilon\tau$ -theorem, so \mathbf{L} proves a disjunction of the form

$$(P(f(t_1)) \rightarrow P(t_1)) \vee \cdots \vee (P(f(t_k)) \rightarrow P(t_k))$$

for some terms t_1, \dots, t_k . (This is a Herbrand disjunction of $\exists z A(z)$.) Each term t_i is of the form $f^i(s)$ for some $i \geq 0$ and a term s which does not start with f . By rearranging the disjuncts to group disjuncts with the same innermost term s together (using commutativity of \vee) and by adding additional disjuncts as needed (using weakening), from this we obtain a formula

$$\begin{aligned} & (P(f^{j_1+1}(s_1)) \rightarrow P(f^{j_1}(s_1))) \vee \cdots \vee (P(f(s_1)) \rightarrow P(s_1)) \vee \\ & \quad \vdots \\ & (P(f^{j_l+1}(s_l)) \rightarrow P(f^{j_l}(s_l))) \vee \cdots \vee (P(f(s_l)) \rightarrow P(s_l)) \end{aligned}$$

Let j be the largest among j_1, \dots, j_l . By uniformly replacing $P(f^i(s_j))$ by A_{j+2-i} in the proof of the last formula and contracting identical disjuncts, we obtain a proof in \mathbf{L} of $(A_1 \rightarrow A_2) \vee \cdots \vee (A_{j+1} \rightarrow A_{j+2})$. This is B_m for $m = j+1$, and since $j \geq 1$, $m \geq 2$. \dashv

A formula of the form B_m is provable in \mathbf{L} iff \mathbf{L} is a finite-valued Gödel logic \mathbf{LC}_n (Proposition 6.5). By contrast, no B_m is provable in intuitionistic logic \mathbf{H} , Jankov logic \mathbf{KC} , or in infinite-valued Gödel logic \mathbf{LC} (Proposition 6.3).

- PROPOSITION 6.3. 1. $\mathbf{LC}_m \vdash B_m$
2. $\mathbf{L} \not\vdash B_n$ for \mathbf{L} any of \mathbf{LC}_m with $m > n$, \mathbf{LC} , \mathbf{KC} , \mathbf{H} .

PROOF. (1) Follows by definition, since $\mathbf{LC}_m = \mathbf{LC} + B_m$.

(2) Let $v(A_i) = 1/i$ if $i < n$ and $v(A_{n+1}) = 0$. This is a valuation in a truth value set with m elements if $m > n$ (i.e., a valuation in the Gödel semantics for \mathbf{LC}_m). It is also a valuation in the infinite truth value set $[0, 1]$ of \mathbf{LC} . For all $i \leq n$, $v(A_i) > v(A_{i+1})$ and hence $v(B_n) < 1$. So B_n is not a tautology of \mathbf{LC}_m or \mathbf{LC} . Since $\mathbf{H} \subsetneq \mathbf{KC} \subsetneq \mathbf{LC}$, the result also follows for \mathbf{KC} and \mathbf{H} . \dashv

PROPOSITION 6.4. $\mathbf{H} + B_m \vdash Lin$

PROOF. Simultaneously substitute A for A_i if i is odd, and B for A_i if i is even in B_m . The result is one of

$$\begin{aligned} & (A \rightarrow B) \vee (B \rightarrow A) \vee \cdots \vee (A \rightarrow B) \\ & (A \rightarrow B) \vee (B \rightarrow A) \vee \cdots \vee (B \rightarrow A) \end{aligned}$$

Both are equivalent in \mathbf{H} to $(A \rightarrow B) \vee (B \rightarrow A)$. \dashv

PROPOSITION 6.5. If $\mathbf{L} \vdash B_n$, then $\mathbf{L} = \mathbf{LC}_m$ for some m .

PROOF. Hosoi [1966] showed that the n -valued Gödel logic is axiomatized by $\mathbf{H} + R_{n-1}$, where R_n is

$$A_1 \vee (A_1 \rightarrow A_2) \vee \cdots \vee (A_{n-1} \rightarrow A_n) \vee \neg A_n.$$

By simultaneously substituting \top for A_1 , and \perp for A_{n+1} , and A_{i-1} for A_i ($i = 2, \dots, n$) in B_n , we obtain

$$(\top \rightarrow A_1) \vee (A_1 \rightarrow A_2) \vee \dots \vee (A_{n-2} \rightarrow A_{n-1}) \vee (A_{n-1} \rightarrow \perp),$$

which is equivalent to R_{n-1} in \mathbf{H} . Hence, since $\mathbf{L} \vdash B_n$, $\mathbf{LC}_n \subseteq \mathbf{L}$.

Furthermore, Hosoi [1967, Lemma 4.1] showed that if $\mathbf{L} \vdash Lin$ then $\mathbf{L} = \mathbf{LC}_m$ for some m or $L = \mathbf{LC}$. Since $\mathbf{L} \vdash B_n$, $\mathbf{L} \vdash Lin$ by Proposition 6.4. The result follows as $\mathbf{LC} \not\vdash B_n$ and so $\mathbf{L} \neq \mathbf{LC}$.⁷ \dashv

COROLLARY 6.6. *No intermediate logic except \mathbf{LC}_m has the extended first $\varepsilon\tau$ -theorem. In particular, intuitionistic logic \mathbf{H} , Jankov logic \mathbf{KC} , and infinite-valued Gödel logic \mathbf{LC} do not have the extended first $\varepsilon\tau$ -theorem.*

We have restricted \mathbf{L} here to be an intermediate propositional logic. However, it bears remarking that Theorem 6.2 does not require that \mathbf{L} contains \mathbf{H} . An inspection of the proof shows that all that is required is that $\mathbf{L} \vdash A \rightarrow A$, and in \mathbf{L} , \vee is provably commutative, associative, and idempotent, and has weakening ($\mathbf{L} \vdash A \rightarrow (A \vee B)$). Thus, Corollary 6.6 applies to any $\varepsilon\tau$ -calculus based on a logic which has these properties (such as, say, Łukasiewicz logic.)

The extended first ε -theorem in classical logic shows that if an existential formula $\exists x A(x)$ is provable, so is a disjunction of instances $\bigvee_i A(t_i)$. Clearly this is equivalent to: if $\forall x A(x) \rightarrow B$ is provable so is $\bigwedge A(t_i) \rightarrow B$. Without the interdefinability of \forall and \exists , the question arises whether the alternative form of the ε -theorem might hold in an intermediate $\varepsilon\tau$ -calculus even if the standard form does not. We'll show that the versions are, in fact, equivalent even in intermediate logics.

PROPOSITION 6.7. *The following are equivalent:*

1. *If $\mathbf{L}\varepsilon\tau \vdash A(e)$ then $\mathbf{L} \vdash \bigvee_i A(t_i)$.*
2. *If $B(e') \vdash_{\mathbf{L}\varepsilon\tau} C$ then $\bigwedge_j B(s_j) \vdash_{\mathbf{L}} C$ for C $\varepsilon\tau$ -free.*
3. *If $B(e') \vdash_{\mathbf{L}\varepsilon\tau} C(e)$ then $\bigwedge_j B(s_j) \vdash_{\mathbf{L}} \bigvee_i C(t_i)$.*

PROOF. (1) implies (3): Suppose

$$B(e') \vdash_{\mathbf{L}\varepsilon\tau} C(e).$$

By the deduction theorem,

$$\vdash_{\mathbf{L}\varepsilon\tau} B(e') \rightarrow C(e).$$

By (1) we have terms s_i, t_i so that

$$\vdash \bigvee_i (B(s_i) \rightarrow C(t_i))$$

⁷In Hosoi's nomenclature, \mathbf{LC}_n is \mathbf{S}_{n-1} and \mathbf{LC} is \mathbf{S}_ω .

By intuitionistic logic,

$$\begin{aligned} & \vdash \bigwedge_i B(s_i) \rightarrow \bigvee_i C(t_i) \text{ and so} \\ & \bigwedge_i B(s_i) \vdash \bigvee_i C(t_i) \end{aligned}$$

by the deduction theorem.

(3) clearly implies (1) and (2).

(2) implies (1): Let X be a propositional variable. $A \vdash_{\mathbf{H}} (A \rightarrow X) \rightarrow X$. So if $\mathbf{L}\varepsilon\tau \vdash A(e)$ then by the deduction theorem,

$$\begin{aligned} & A(e) \rightarrow X \vdash_{\mathbf{L}\varepsilon\tau} X \text{ and by (2),} \\ & \bigwedge_i (A(s_i) \rightarrow X) \vdash_{\mathbf{L}} X. \end{aligned}$$

Now substitute $\bigvee_i A(s_i)$ for X :

$$\bigwedge_i (A(s_i) \rightarrow \bigvee_i A(s_i)) \vdash_{\mathbf{L}} \bigvee_i A(s_i).$$

The formula on the left is provable intuitionistically. ◻

§7. Elimination Sets and Excluded Middle. The basic idea of Hilbert's proof of the extended first ε -theorem is this: Suppose we have a proof of $E \equiv D(e)$ from critical formulas $\Gamma, \Lambda(e)$, where e is a critical ε -term and $\Lambda(e)$ is a set of critical formulas belonging to e . Now we find terms t_1, \dots, t_k such that replacing e by t_i allows us to remove the critical formulas $\Lambda(e)$, while at the same time replacing the end-formula $D(e)$ by $\bigvee_{i=1}^k D(t_i)$ and the remaining critical formulas by $\Gamma[t_1/e], \dots, \Gamma[t_k/e]$. We repeat this procedure in such a way that eventually all critical formulas are removed and we are left with a disjunction of instances of E , as required by the first ε -theorem. The difficulty of making this work lies in three challenges. The first is to find a suitable way of selecting ε -terms e and corresponding critical formulas $\Lambda(e)$ so that the $\Lambda(e)$ can be removed. The second is to ensure that in passing from Γ to $\Gamma[t_i/e]$ we again obtain critical formulas.⁸ The third challenge is to guarantee that the process eventually terminates with no critical formulas remaining.

In this section we address the first challenge by considering the condition that suffices to overcome it: the existence of complete e -elimination sets (defined below) for every $\varepsilon\tau$ -term e . We then show why this condition is satisfied in classical logic, so we can clarify the role of excluded middle in the proof for the classical case, as well as how the proof for classical logic and those for intermediate logics given later correspond to one another. We will discuss the condition for intermediate logics in Section 8 and the remaining challenges in Section 9.

⁸Replacing an ε -term in a critical formula by another term does in general not result in a critical formula. E.g., let $A(y) \equiv B(\varepsilon_x C(x, y), y)$ and $e \equiv \varepsilon_x C(x, t)$ then $A(t)[s/e]$ is $B(s, t)$ but $A(\varepsilon_y A(y))[s/e]$ is just $A(\varepsilon_y A(y))$.

DEFINITION 7.1. Suppose $\Gamma \vdash_{\mathbf{L}\varepsilon\tau}^{\tau} D$ with critical formulas Γ , and e is an ε -term $\varepsilon_x A(x)$ (τ -term $\tau_x A(x)$). If $C \equiv A(t) \rightarrow A(\varepsilon_x A(x)) \in \Gamma$ ($C \equiv A(\tau_x A(x)) \rightarrow A(t) \in \Gamma$) we say e is *the critical $\varepsilon\tau$ -term of C* , that e *belongs to C* , and that e is a *critical $\varepsilon\tau$ -term of π* .

DEFINITION 7.2. Suppose $\Gamma, \Lambda(e), \Lambda'(e) \vdash_{\mathbf{L}\varepsilon\tau}^{\tau} D(e)$ where $\Lambda(e) \cup \Lambda'(e)$ are all critical formulas belonging to e . A set of terms s_1, \dots, s_k is an *e -elimination set for π and $\Lambda(e)$* if

$$\Gamma[s_1/e], \dots, \Gamma[s_k/e], \Lambda'(e) \vdash_{\mathbf{L}} D(s_1) \vee \dots \vee D(s_k).$$

If $\Lambda(e)$ is the set of all critical formulas belonging to e (i.e., $\Lambda'(e) = \emptyset$) then an e -elimination set for $\Lambda(e)$ is called a *complete e -elimination set*.

Here, $\Gamma[s_i/e]$ means the result of replacing, in each formula in Γ , every occurrence of e by s_i . If $T = \{s_1, \dots, s_k\}$ we write $\Gamma[T]$ for $\Gamma[s_1], \dots, \Gamma[s_k]$. Note that we do not require in the definition of e -elimination sets that the formulas in $\Gamma[s_i/e]$ are actually critical formulas.

LEMMA 7.3. *If $C \equiv A(t) \rightarrow A(e)$ or $C \equiv A(e) \rightarrow A(t)$ is a critical formula with critical $\varepsilon\tau$ -term e , then $C[s/e]$ is $A(t[s/e]) \rightarrow A(s)$ or $A(s) \rightarrow A(t[s/e])$, respectively.*

PROOF. Since e is the critical ε -term of C , $e \equiv \varepsilon_x A(x)$ or $e \equiv \tau_x A(x)$. Hence, e cannot occur in $A(x)$, since otherwise it would be a proper subterm of itself. \dashv

LEMMA 7.4. *If $\Gamma \vdash_{\mathbf{L}} D$ then $\Gamma[t/e] \vdash_{\mathbf{L}} D[t/e]$*

PROOF. Any proof of D from Γ using modus ponens and axioms of \mathbf{L} remains correct if terms in it are uniformly replaced by other terms. \dashv

LEMMA 7.5. *In any intermediate logic \mathbf{L} :*

1. *If $\Gamma, A \vdash C$ and $\Gamma', B \vdash_{\mathbf{L}} D$ then $\Gamma, \Gamma', A \vee B \vdash C \vee D$.*
2. *If $\Gamma, A \vdash_{\mathbf{L}} C$ and $B \vdash A$, then $\Gamma, B \vdash C$.*

We are now in a position to apply the preceding lemmas and the concept of elimination sets to the case of classical logic. This elucidates how the first challenge is solved in the proof of the extended first $\varepsilon\tau$ -theorem for classical logic where τ -terms and corresponding critical formulas may also be present. (For Hilbert's original proof for the ε -calculus without τ -terms, see Hilbert and Bernays [1939] or Moser and Zach [2006].)

PROPOSITION 7.6. *In $\mathbf{C}\varepsilon\tau$, every critical formula $C(e)$ has an e -elimination set.*

PROOF. Suppose first that e is an ε -term; then $C(e)$ is $A(s) \rightarrow A(e)$. Let $\Lambda'(e)$ be the critical formulas belonging to e other than $C(e)$, and Γ the remaining critical formulas for which e is not critical. So we have:

$$\Gamma, \Lambda'(e), A(s) \rightarrow A(e) \vdash_{\mathbf{C}} D(e)$$

On the one hand, by replacing e everywhere by s we get

$$\Gamma[s/e], \Lambda'(s), A(s[s/e]) \rightarrow A(s) \vdash_{\mathbf{C}} D(s)$$

and by Lemma 7.5(2), since $A(s) \vdash A(t[s/e]) \rightarrow A(s) \in \Lambda'(s)$ and $A(s) \vdash C(s)$,

$$\Gamma[s/e], A(s) \vdash_{\mathbf{C}} D(s)$$

On the other hand, since $\neg A(s) \vdash A(s) \rightarrow A(e)$,

$$\Gamma, \neg A(s) \vdash_{\mathbf{C}} D(e) \text{ and so,}$$

$$\Gamma, \Gamma[s/e], A(s) \vee \neg A(s) \vdash_{\mathbf{C}} D(e) \vee D(s).$$

by Lemma 7.5(1). Since $\mathbf{C} \vdash A(s) \vee \neg A(s)$ we have

$$\Gamma, \Gamma[s/e] \vdash_{\mathbf{C}} D(e) \vee D(s)$$

Thus, $\{e, s\}$ is an e -elimination set for the critical formula $C(e)$.

Similarly, if e is a τ -term and $C(e)$ is $A(e) \rightarrow A(s)$ we get

$$\Gamma[s/e], \Lambda'(s), C(s) \vdash_{\mathbf{C}} D(s)$$

and by Lemma 7.5(2), since $\neg A(s) \vdash C' \in \Lambda'(e)$ and $\neg A(s) \vdash C(s)$,

$$\Gamma[s/e], \neg A(s) \vdash_{\mathbf{C}} D(s)$$

On the other hand, $A(s) \vdash C(e)$, so

$$\Gamma, A(s) \vdash_{\mathbf{C}} D(e) \text{ and}$$

$$\Gamma, \Gamma[s/e], \neg A(s) \vee A(s) \vdash_{\mathbf{C}} D(e) \vee D(s).$$

by Lemma 7.5(1). ⊢

More generally, the set of all critical formulas belonging to e has a (complete) e -elimination set in \mathbf{C} :

PROPOSITION 7.7. *In $\mathbf{C}_{\varepsilon\tau}$, every critical $\varepsilon\tau$ -term has a complete e -elimination set.*

PROOF. Let $C_1 \equiv A(s_1) \rightarrow A(e)$, \dots , $C_k \equiv A(s_k) \rightarrow A(e)$ be the critical formulas belonging to e if e is an ε -term. Since

$$\Gamma, C_1(e), \dots, C_k(e) \vdash_{\mathbf{C}} D(e), \text{ also}$$

$$\Gamma[s_i/e], C_1(s_i), \dots, C_k(s_i) \vdash_{\mathbf{C}} D(s_i)$$

(writing $C_j(s_i)$ for $C_j[s_i/e]$). Since $A(s_i) \vdash A(s_j(s_i)) \rightarrow A(s_i) \equiv C_j(s_i)$,

$$\Gamma[s_i/e], A(s_i) \vdash_{\mathbf{C}} D(s_i)$$

by Lemma 7.5(2). By applying Lemma 7.5(1),

$$\Gamma[s_1/e], \dots, \Gamma[s_k/e], A(s_1) \vee \dots \vee A(s_k) \vdash_{\mathbf{C}} D(s_1) \vee \dots \vee D(s_k).$$

On the other hand, since $\neg A(s_i) \vdash A(s_i) \rightarrow A(e)$, we get $\neg A(s_1) \wedge \dots \wedge A(s_k) \vdash C_j(e)$ for each $j = 1, \dots, k$, so we also have, from the first line by Lemma 7.5(2),

$$\Gamma, \neg A(s_1) \wedge \dots \wedge \neg A(s_k) \vdash_{\mathbf{C}} D(e).$$

Since $\bigvee_i A(s_i) \vee (\bigwedge_i \neg A(s_i))$ is an instance of excluded middle, we have

$$\Gamma, \Gamma[s_1/e], \dots, \Gamma[s_k/e] \vdash_{\mathbf{C}} D(e) \vee D(s_1) \vee \dots \vee D(s_k).$$

If e is a τ -term, then the critical formulas are of the form $C_j(e) \equiv A(e) \rightarrow A(s_j(e))$ and consequently $C_j(s_i)$ is $A(s_i) \rightarrow A(s_j(s_i))$. Each is implied by $\neg A(s_i)$, so we have

$$\Gamma[s_1/e], \dots, \Gamma[s_k, e], \neg A(s_1) \vee \dots \vee \neg A(s_k) \vdash_{\mathbf{C}} D(s_1) \vee \dots \vee D(s_k)$$

On the other hand, $A(s_1) \wedge \dots \wedge A(s_k) \vdash A(e) \rightarrow A(s_j)$, so

$$\Gamma, A(s_1) \wedge \dots \wedge A(s_k) \vdash_{\mathbf{C}} D(e)$$

and consequently

$$\Gamma, \Gamma[s_1/e], \dots, \Gamma[s_k/e] \vdash_{\mathbf{C}} D(e) \vee D(s_1) \vee \dots \vee D(s_k),$$

since $\bigwedge_i A(s_i) \vee \bigvee_i \neg A(s_i)$ is a tautology.

In each case, $\{e, s_1, \dots, s_k\}$ is an e -elimination set. \dashv

Remark 7.8. Of course, the fact that in \mathbf{C} we have complete e -elimination sets can also be obtained by applying Proposition 7.6 k -many times. Applying it to $C_1(e)$ results in $T_1 = \{e, s_1(e)\}$, applying it to $C_2(e)$ in $T_2 = \{e, s_1(e), s_2(e), s_1(s_2(e))\}$, to C_3 in $T_3 = \{e, s_1(e), s_2(e), s_1(s_2(e)), s_3(e), s_1(s_3(e)), s_2(s_3(e)), s_1(s_2(s_3(e)))\}$, etc., i.e., the resulting disjunction has 2^{k+1} disjuncts, whereas the disjunction resulting from Proposition 7.7 only has $k+1$ disjuncts. However, see Remark 9.7.

We know that intermediate logics other than \mathbf{LC}_m do not have the extended first $\varepsilon\tau$ -theorem and so not every $\varepsilon\tau$ -term will have complete e -elimination sets. However, if the starting formula E is of a special form, they sometimes do. In the proof for the classical case above, this required excluded middle. But it need not. For instance, if E is negated, then weak excluded middle ($\neg A \vee \neg\neg A$) is enough.

PROPOSITION 7.9. *If $\mathbf{L} \vdash J$, then every $\varepsilon\tau$ -term in an $\mathbf{LE}\varepsilon\tau$ -proof of $\bigvee_j \neg D_j$ has a complete e -elimination set.*

PROOF. Let $C_1 \equiv A(s_1) \rightarrow A(e)$, \dots , $C_k \equiv A(s_k) \rightarrow A(e)$ be the critical formulas belonging to e if e is an ε -term. As before, we have

$$\Gamma[s_i/e], A(s_i) \vdash_{\mathbf{L}} \bigvee_j \neg D_j(s_i)$$

In \mathbf{KC} , $B \rightarrow (\neg C_1 \vee \neg C_2) \vdash \neg\neg B \rightarrow (\neg C_1 \vee \neg C_2)$, so

$$\Gamma[s_i/e], \neg\neg A(s_i) \vdash_{\mathbf{L}} \bigvee_j \neg D_j(s_i)$$

We obtain

$$\begin{aligned} & \Gamma[s_1/e], \dots, \Gamma[s_k/e], \\ \neg\neg A(s_1) \vee \dots \vee \neg\neg A(s_k) \vdash_{\mathbf{L}} & \bigvee_j \neg D_j(s_1) \vee \dots \vee \bigvee_j \neg D_j(s_k) \end{aligned}$$

Again as before, we have

$$\Gamma, \neg A(s_1) \wedge \dots \wedge \neg A(s_k) \vdash_{\mathbf{L}} \bigvee_j \neg D_j(e),$$

and together

$$\begin{aligned} & \Gamma[s_1/e], \dots, \Gamma[s_k/e], \Gamma, \\ (\neg A(s_1) \wedge \dots \wedge \neg A(s_k)) \vee & \\ \neg\neg A(s_1) \vee \dots \vee \neg\neg A(s_k) \vdash_{\mathbf{L}} & \bigvee_j \neg D_j(e) \vee \bigvee_j \neg D_j(s_1) \vee \dots \vee \bigvee_j \neg D_j(s_k) \end{aligned}$$

Since

$$(\neg A(s_1) \wedge \dots \wedge \neg A(s_k)) \vee \neg\neg A(s_1) \vee \dots \vee \neg\neg A(s_k)$$

is provable from weak excluded middle, the claim is proved. \dashv

§8. Elimination Sets using B_m and Lin . We've showed in Theorem 6.2 that the provability of B_m for some $m \geq 2$ is a necessary condition for an intermediate logic to have the extended first $\varepsilon\tau$ -theorem. In this section, we show that it is also sufficient: if $\mathbf{L} \vdash B_m$ for some m , then every critical $\varepsilon\tau$ -term has a complete e -elimination set.

We use the notion of e -elimination sets developed in Section 7: the existence of an e -elimination set for a set of critical formulas Λ guarantees that these critical formulas can be removed from a proof, while the end-formula is replaced by a disjunction of instances of the original end-formula. The proofs of the existence of e -elimination sets proceed by replacing an $\varepsilon\tau$ -term e in such a way that the disjunction of formulas Λ' resulting from such replacements become (provable from) tautologies in the underlying propositional logic. In Proposition 7.6, we showed how to do this for a single critical formula $A(s) \rightarrow A(e)$ in classical logic: We replace e first by s resulting in $A(s[s/e]) \rightarrow A(s) \vdash D(s)$, and then by itself (i.e., no replacement), resulting in $A(s) \rightarrow A(e) \vdash D(e)$. This gives

$$(A(s[s/e]) \rightarrow A(s)) \vee (A(s) \rightarrow A(e)) \vdash D(s) \vee D(e),$$

but the formula on the left is a classical tautology. When eliminating multiple critical formulas at the same time (e.g., all critical formulas belonging to a single $\varepsilon\tau$ -term), the resulting tautologies are more complicated. In the original proof, they are all equivalent to excluded middle, and so the proofs do not apply to intermediate propositional logics. Below, we show how this can nevertheless be

done as long as the underlying propositional logic contains B_m . We have to distinguish two kinds of critical formulas:

DEFINITION 8.1. A critical formula $A(s) \rightarrow A(e)$ (or $A(e) \rightarrow A(s)$ if e is a τ -term) is called *predicative* if e does not occur in s , and *impredicative* otherwise.⁹

If all critical formulas are predicative, $\mathbf{L} \vdash \text{Lin}$ suffices (Lemma 8.5). If $\mathbf{L} \vdash B_m$ for some m , we can eliminate the impredicative critical formulas for some $\varepsilon\tau$ -term e in a similar way: by successively replacing e by suitable terms, we obtain a proof of a disjunction of instances of $D(e)$ from a formula provable from B_m . The number m determines the number of necessary replacements. Once impredicative critical formulas are removed, we can use Lemma 8.5 to remove the remaining predicative critical formulas.

We prove the result for impredicative critical formulas first (Lemma 8.3). In preparation for the proof we first consider an example to illustrate the basic idea. Suppose $\mathbf{LC}_3\varepsilon\tau \vdash D(e)$ with the set of critical formulas

$$\begin{aligned} C_s(e) &\equiv A(s(e)) \rightarrow A(e) \\ C_t(e) &\equiv A(t(e)) \rightarrow A(e) \\ C_u(e) &\equiv A(u) \rightarrow A(e) \\ C_v(e) &\equiv A(v) \rightarrow A(e) \end{aligned}$$

where terms u and v do not contain e . Let $C(e)$ be the conjunction of these critical formulas. We have $C_s(e), C_t(e), C_u(e), C_v(e) \vdash_{\mathbf{LC}_3} D(e)$.

We'll consider sets of terms X_i where $X_0 = \{e\}$ and $X_{i+1} = \{s(x), t(x) : x \in X_i\}$. For the sake of readability we will leave out parentheses then writing these terms, e.g., $s(t(e))$ is abbreviated as ste .

For every $w \in X_1$ we have $C(w) \vdash D(w)$. So, by applying Lemma 7.5(1) twice, we get:

$$\begin{aligned} &C(se) \vee C_s(e), \\ &C(te) \vee C_t(e), \\ &C_u(e), C_v(e) \vdash_{\mathbf{LC}_3} D(e) \vee D(se) \vee D(te) \end{aligned}$$

By distributivity, $C(se) \vee C_s(e)$ is equivalent to

$$\begin{aligned} &(C_s(se) \vee C_s(e)) \wedge (C_t(se) \vee C_s(e)) \wedge \\ &(C_u(se) \vee C_s(e)) \wedge (C_v(se) \vee C_s(e)) \end{aligned}$$

⁹The terminology is chosen in analogy to the notion of predicative definition, in which the definiens does not itself involve (quantification over) the thing being defined. Likewise here, the ‘‘definition’’ $A(s)$ of the ε -term e does not mention the ε -term e it defines if the critical formula is predicative. Such restrictions of definitions (and instances of comprehension) are the basis of predicative mathematics, which goes back to Weyl and Russell. There is, however, no deeper connection between our choice of terminology and predicative mathematics.

Thus we get

$$\begin{aligned} & C_s(se) \vee C_s(e), C_t(se) \vee C_s(e), \\ & C_u(se) \vee C_s(e), C_v(se) \vee C_s(e), \\ & C(te) \vee C_t(e), \\ & C_u(e), C_v(e) \vdash_{\mathbf{LC}_3} D(e) \vee D(se) \vee D(te). \end{aligned}$$

We have

$C_u(e) \equiv A(u) \rightarrow A(e) \vdash_{\mathbf{LC}_3} (A(u) \rightarrow A(se)) \vee (A(se) \rightarrow A(e)) \equiv C_u(se) \vee C_s(e)$ using *Lin*. Similarly, $C_v(e) \vdash_{\mathbf{LC}_3} C_v(se) \vee C_s(e)$, so we have by Lemma 7.5(2):

$$\begin{aligned} & C_s(se) \vee C_s(e), C_t(se) \vee C_s(e), \\ & C(te) \vee C_t(e), \\ & C_u(e), C_v(e) \vdash_{\mathbf{LC}_3} D(e) \vee D(se) \vee D(te) \end{aligned}$$

Repeating this consideration with $C(te) \vee C_t(e)$ yields

$$\begin{aligned} & C_s(se) \vee C_s(e), C_t(se) \vee C_s(e), \\ & C_s(te) \vee C_t(e), C_t(te) \vee C_t(e), \\ & C_u(e), C_v(e) \vdash_{\mathbf{LC}_3} D(e) \vee D(se) \vee D(te) \end{aligned}$$

Note that each of the four resulting disjunctions has as first disjunct a substitution instance of a critical formula of the form $C_s(w)$ or $C_t(w)$ where $w \in X_1$. X_2 are the terms of the form $s(w)$ and $t(w)$. So we can repeat the process, pairing $C(s(w))$ with $C_s(w)$ and $C(t(w))$ with $C_t(w)$, i.e., obtaining $C(sse) \vee C_s(se) \vee C_s(e)$, $C(tse) \vee C_t(se) \vee C_s(e)$, etc. In each case, after distributing and removing conjuncts of the form $C_u(w) \vee \dots$ we are left with now eight disjunctions:

$$\begin{aligned} & C_s(sse) \vee C_s(se) \vee C_s(e), \\ & C_t(sse) \vee C_s(se) \vee C_s(e), \\ & \vdots \\ & C_t(tte) \vee C_t(te) \vee C_t(e), \\ & C_u(e), C_v(e) \vdash_{\mathbf{LC}_3} D(e) \vee D(se) \vee D(te) \vee D(sse) \vee \dots \vee D(tte) \end{aligned}$$

It remains to show that the formulas on the right of the turnstile are provable in \mathbf{LC}_3 . First, consider a formula of the form $C_i(w) \vee \dots \vee C_j(e)$, e.g.,

$$\begin{aligned} & C_s(tse) \vee C_t(se) \vee C_s(e), \text{ i.e.,} \\ & (A(stse)) \rightarrow A(tse) \vee (A(tse) \rightarrow A(se)) \vee (A(se) \rightarrow A(e)) \end{aligned}$$

In each disjunct, the consequent equals the antecedent of the disjunct immediately to the right, i.e., it is a substitution instance of

$$(A_1 \rightarrow A_2) \vee (A_2 \rightarrow A_3) \vee (A_3 \rightarrow A_4)$$

i.e., of B_3 . Since $\mathbf{LC}_3 \vdash B_3$, these are all provable.

If we take $D'(e)$ to be the disjunction obtained on the right, we have

$$\begin{aligned} C_u(e), C_v(e) &\vdash_{\mathbf{LC}_3} D'(e) \text{ and thus also} \\ C_u(u), C_v(u) &\vdash_{\mathbf{LC}_3} D'(u) \text{ and} \\ C_u(v), C_v(u) &\vdash_{\mathbf{LC}_3} D'(v). \end{aligned}$$

As $C_u(u)$ and $C_v(v)$ are of the form $A \rightarrow A$ this reduces to

$$\begin{aligned} C_u(v) &\vdash_{\mathbf{LC}_3} D'(u) \text{ and} \\ C_v(u) &\vdash_{\mathbf{LC}_3} D'(v) \text{ and therefore:} \\ C_u(v) \vee C_v(u) &\vdash_{\mathbf{LC}_3} D'(u) \vee D'(v). \end{aligned}$$

But $C_u(v) \vee C_v(u)$ is an instance of Lin .

LEMMA 8.2. *If $\mathbf{L} \vdash Lin$, then for all m ,*

$$A_1 \rightarrow A_{m+1} \vdash_{\mathbf{L}} (A_1 \rightarrow A_2) \vee \cdots \vee (A_m \rightarrow A_{m+1}).$$

PROOF. By induction on m . If $m = 1$, this amounts to the claim: $A_1 \rightarrow A_2 \vdash_{\mathbf{L}} A_1 \rightarrow A_2$, which is trivial. Now suppose the claim holds for m . Then

$$A_1 \rightarrow A_{m+2} \vdash_{\mathbf{L}} (A_1 \rightarrow A_{m+1}) \vee (A_{m+1} \rightarrow A_{m+2})$$

from the instance $(A_1 \rightarrow A_{m+1}) \vee (A_{m+1} \rightarrow A_1)$ of Lin . By induction hypothesis,

$$A_1 \rightarrow A_{m+1} \vdash_{\mathbf{L}} (A_1 \rightarrow A_2) \vee \cdots \vee (A_m \rightarrow A_{m+1})$$

and the claim follows by **H**. ◻

LEMMA 8.3. *Suppose $\mathbf{L} \vdash B_m$. If $\Delta(e)$ are the impredicative critical formulas belonging to the $\varepsilon\tau$ -term e , then $\Delta(e)$ has an e -elimination set.*

PROOF. Suppose $\Gamma, \Delta(e), \Pi(e) \vdash_{\mathbf{L}} D(e)$, where $\Pi(e)$ is the set of predicative critical formulas belonging to e .

Suppose $\Pi(e)$ and $\Delta(e)$ consist of, respectively, the critical formulas

$$\begin{aligned} C_{u_i}(e) &\equiv A(u_i) \rightarrow A(e) \\ C_{s_i}(e) &\equiv A(s_i(e)) \rightarrow A(e). \end{aligned}$$

The proof generalizes the preceding example: we successively substitute terms for e in such a way that a disjunction of instances of D is implied by substitution instances of the critical formulas in Γ together with a disjunction of the form B_m plus the predicative critical formulas $\Pi(e)$. Once $k = m$, the disjunction becomes provable from B_m .

Let $T_0 = \{e\}$ and $T_{i+1} = \{s_j(t) : t \in T_i, j \leq r\}$. Let $\Gamma(T) = \{C[t/e] : C \in \Gamma, t \in T\}$. If w is a word over $\{s_1, \dots, s_r\}$, i.e., $w = s_{i_1} \dots s_{i_k}$ then we write $w_j(t)$ for $s_{i_j}(s_{i_{j+1}}(\dots s_{i_k}(t) \dots))$. So e.g., if $w = sst$ then $w_1(e)$ is $s(s(t(e)))$, $w_3(e) = t(e)$ and $w_4(e) = e$. Let $W(m)$ be the set of all length m words over s_1, \dots, s_r .

We show by induction on m that

$$\Gamma, \Gamma(T_1), \dots, \Gamma(T_{m-1}), \Lambda_m, \Pi(e) \vdash_{\mathbf{L}} \bigvee_{i=1}^m \bigvee_{t \in T_{i-1}} D(t)$$

where

$$\begin{aligned} \Lambda_i &= \left\{ \bigvee_{i=1}^m C(w, i) : w \in W(m) \right\} \text{ and} \\ C(w, i) &\equiv A(w_i(e)) \rightarrow A(w_{i+1}(e)). \end{aligned}$$

The induction basis is $m = 1$. Then

$$\begin{aligned} T_1 &= \{s_1(e), \dots, s_r(e)\}, \\ W(1) &= \{s_1, \dots, s_r\}, \\ C(s_j, 1) &\equiv A(s_j(e)) \rightarrow A(e) \end{aligned}$$

and so each disjunction in Λ_i is just one of the impredicative critical formulas $A(s_j(e)) \rightarrow A(e)$, i.e., $\Lambda_1 = \Delta(e)$. Likewise, the disjunction on the right is just $D(e)$. So the claim holds by the assumption that $\Gamma, \Delta(e), \Pi(e) \vdash D(e)$.

Now let $v = s_{i_1} \dots s_{i_m}$ be a length m word, and

$$\begin{aligned} \Lambda_m(v) &= \left\{ \bigvee_{i=1}^m (A(w_i(e)) \rightarrow A(w_{i+1}(e))) : w \in W(m) \setminus \{v\} \right\} \\ C(v) &\equiv \bigvee_{i=1}^m (A(v_i(e)) \rightarrow A(v_{i+1}(e))) \end{aligned}$$

In other words, $\Lambda_m = \Lambda_m(v) \cup \{C(v)\}$. We'll abbreviate $\Gamma(T_1), \dots, \Gamma(T_{m-1})$ as Γ' , and $\bigvee_{i=1}^m \bigvee_{t \in T_{i-1}} D(t)$ as D' . The induction hypothesis can then be written as:

$$\Gamma, \Gamma', C(v), \Lambda_m(v), \Pi(e) \vdash_{\mathbf{L}} D'$$

Take $t = v_1(e)$ i.e., $s_{i_1}(\dots s_{i_m}(e))$. By replacing e by t in π , we have

$$\begin{aligned} \Gamma(t), \Lambda(t), \Pi(t) &\vdash_{\mathbf{L}} D(t) \text{ and so also} \\ \Gamma(t), \bigwedge \Lambda(t) \wedge \bigwedge \Pi(t) &\vdash_{\mathbf{L}} D(t) \end{aligned}$$

Combining this with the induction hypothesis using Lemma 7.5(1) we have

$$\begin{aligned} &\Gamma, \Gamma', \Gamma(t), \\ &(\bigwedge \Lambda(t) \wedge \bigwedge \Pi(t)) \vee C(v), \\ &\Lambda_m(v), \Pi(e) \vdash_{\mathbf{L}} D' \vee D(t) \end{aligned}$$

If we write $\Xi \vee G$ for $\{F \vee G : F \in \Xi\}$, by distributivity,

$$\begin{aligned} & \Gamma, \Gamma', \Gamma(t), \\ & \Lambda(t) \vee C(v), \\ & \Pi(t) \vee C(v), \\ & \Lambda_m(v), \Pi(e) \vdash_{\mathbf{L}} D' \vee D(t) \end{aligned}$$

Recall that $t \equiv v_1(e)$ where v is a word of length m . The formulas in $\Pi(t)$ are of the form $A(u_i) \rightarrow A(v_1(e))$, so a formula in $\Pi(t) \vee C(v)$ is of the form

$$(A(u_i) \rightarrow A(v_1(e))) \vee (A(v_1(e)) \rightarrow A(v_2(e))) \vee \cdots \vee (A(v_m(e)) \rightarrow A(e)).$$

Every such formula is implied by $A(u_i) \rightarrow A(e)$ by Lemma 8.2, since $\mathbf{H} + B_m \vdash Lin$ by Proposition 6.4. Since $A(u_i) \rightarrow A(e)$ is in $\Pi(e)$, we get:

$$\Gamma, \Gamma', \Gamma(t), \Lambda(t) \vee C(v), \Lambda_m(v), \Pi(e) \vdash_{\mathbf{L}} D' \vee D(t)$$

Every formula in $\Lambda(t) \vee C(v)$ is of the form B_m , specifically,

$$(A(s_i(v_1(e))) \rightarrow A(v_1(e))) \vee (A(v_1(e)) \rightarrow A(v_2(e))) \vee \cdots \vee (A(v_m(e)) \rightarrow A(e)).$$

Since for each $i \leq r$, $A(s_i(v_1(e))) \rightarrow A(v_1(e)) \in \Lambda(t)$, $\Lambda(t) \vee C(v)$ is the set of all disjunctions

$$\bigvee_{i=1}^{m+1} (A(w_i(e)) \rightarrow A(w_{i+1}(e)))$$

where $w = s_i v$ for some $i \leq r$. As every length $m + 1$ word is of this form for some length m word v , repeating this process for all length m words v thus yields

$$\Gamma, \Gamma', \Gamma(T_m), \Lambda_{m+1}, \Pi(e) \vdash_{\mathbf{L}} D' \vee \bigvee_{t \in T_m} D(t)$$

As we've seen, a formula in Λ_m is of the form B_m , so in $\mathbf{L} + B_m$, we have

$$\Gamma, \Gamma(T_1), \dots, \Gamma(T_{m-1}), \Pi(e) \vdash_{\mathbf{L}} \bigvee_{i=1}^m \bigvee_{t \in T_{i-1}} D(t)$$

Thus, the claim follows by taking $T = \{e\} \cup T_{m-1}$.

If e is a τ -term, the proof proceeds analogously. The resulting formulas in $\Lambda(e)$ are then of the form $(A_m \rightarrow A_{m+1}) \vee \cdots \vee (A_1 \rightarrow A_2)$ which is equivalent to B_m . \dashv

LEMMA 8.4. *If $\mathbf{L} \vdash Lin$,*

1. $\mathbf{L} \vdash \bigvee_{j=1}^m \bigwedge_{i=1}^m (A_i \rightarrow A_j)$
2. $\mathbf{L} \vdash \bigvee_{j=1}^m \bigwedge_{i=1}^m (A_j \rightarrow A_i)$

PROOF. As propositional infinite-valued Gödel logic $\mathbf{G}_{\mathbb{R}}$ is axiomatized by $\mathbf{H} + Lin$, it suffices to show that the formulas are valid in the Gödel logic based on the truth value set $[0, 1]$. For (1), in any given valuation, one of the A_j must be maximal, i.e., $A_i \rightarrow A_j$ has value 1 in it for all i . For (2), one of the A_j must be minimal. \dashv

LEMMA 8.5. *Suppose $\mathbf{L} \vdash \text{Lin}$ and $\Gamma, \Pi(e) \vdash_{\mathbf{L}} D(e)$, where $\Pi(e)$ are the predicative critical formulas $A(u_j) \rightarrow A(e)$ belonging to e , there are no impredicative critical formulas belonging to e , and Γ are critical formulas for which e is not critical. Then $\Pi(e)$ has a complete e -elimination set.*

PROOF. First suppose e is an ε -term. Replacing e by u_j results in a proof showing

$$\Gamma[u_j/e], \bigwedge_{i=1}^p (A(u_i) \rightarrow A(u_j)) \vdash_{\mathbf{L}} D(u_j)$$

By applying Lemma 7.5(1), we get

$$\bigcup_j \Gamma[u_j/e], \bigvee_{j=1}^p \bigwedge_{i=1}^p (A(u_i) \rightarrow A(u_j)) \vdash_{\mathbf{L}} \bigvee_j D(u_j)$$

The disjunction of conjunctions on the left is provable in $\mathbf{L} + \text{Lin}$ by Lemma 8.4(1).

If e is a τ -term, we get

$$\bigcup_j \Gamma[u_j/e], \bigvee_{j=1}^p \bigwedge_{i=1}^p (A(u_j) \rightarrow A(u_i)) \vdash_{\mathbf{L}} \bigvee_j D(u_j)$$

and the claim follows by Lemma 8.4(2). \dashv

THEOREM 8.6. *If $\mathbf{L} \vdash B_m$ for some m , then $\mathbf{L}\varepsilon\tau$ has complete e -elimination sets.*

PROOF. Suppose $\Gamma, \Pi(e), \Delta(e) \vdash_{\mathbf{L}} D(e)$, where $\Pi(e)$ are the predicative critical formulas belonging to e , $\Delta(e)$ the impredicative formulas belonging to e , and Γ are critical formulas for which e is not critical. By Lemma 8.3, $\Gamma[T], \Pi(e) \vdash_{\mathbf{L}} \bigvee_{t \in T} D(t)$ where $T = \{e\} \cup T_{m-1}$. Since $\mathbf{L} + B_m \vdash \text{Lin}$ by Proposition 6.4, Lemma 8.5 applies and so $\bigcup_j T[u_j/e]$ is a complete e -elimination set. \dashv

Remark 8.7. The proof of Proposition 7.7 provides essentially Hilbert's way of computing e -elimination sets using excluded middle. However, instead of excluded middle $A \vee \neg A$, classical logic can also be axiomatized over \mathbf{H} by B_2 , i.e., $(A \rightarrow B) \vee (B \rightarrow C)$. The method of computing e -elimination sets using Lemmas 8.3 and 8.5 applied to $\mathbf{LC}_2 = \mathbf{C}$ provides a method for computing e -elimination sets (and hence of Herbrand disjunctions) different from Hilbert's method.

§9. The Hilbert-Bernays Elimination Procedure. Recall that the challenges in the proof of the extended first $\varepsilon\tau$ -theorem include, in addition to the existence of complete elimination sets, guarantees that the new sets $\Gamma[s_i/e]$ are in fact critical formulas (so eliminating a set $\Lambda(e)$ of critical formulas yields a correct $\mathbf{L}\varepsilon\tau$ -proof), and that the process eventually terminates. In Hilbert and Bernays's original proof of the first ε -theorem, this was ensured by processing

sets of critical formulas in a specific order. We briefly review this proof, concentrating on its structure, since we'll apply the same method to critical formulas in $\varepsilon\tau$ -proofs for intermediate logics.

DEFINITION 9.1. Suppose $\vdash_{\mathbf{L}\varepsilon\tau}^{\pi_1} D_1$. A sequence $\langle \Lambda_1(e_1), T_1 \rangle, \dots, \langle \Lambda_k(e_k), T_k \rangle$ is an $\varepsilon\tau$ -elimination sequence iff, for each i ,

1. $\Lambda_i(e_i)$ is a set of critical formulas belonging to e_i ,
2. $\Gamma_i, \Lambda'(e), \Lambda_i(e_i) \vdash_{\mathbf{L}\varepsilon\tau}^{\pi_i} D_i$, where $\Lambda'(e)$ are the critical formulas for e_i not in $\Lambda_i(e_i)$, and Γ_i the remaining critical formulas in π_i ,
3. T_i is an e_i -elimination set for π_i and $\Lambda_i(e_i)$
4. $\Gamma_{i+1} = \Gamma_i[T_i/e_i]$ and $D_{i+1} = \bigvee_{t \in T_i} D_i[t/e_i]$,
5. $\Gamma_{i+1} \vdash_{\mathbf{L}\varepsilon\tau} D_{i+1}$,

and $\Gamma_{k+1} = \emptyset$, that is, $\mathbf{L} \vdash^{\pi_{k+1}} D_{k+1}$. If all $\Lambda'(e_i) = \emptyset$ (i.e., T_i is a complete elimination set for e_i), we say the sequence is a complete elimination sequence.

An $\varepsilon\tau$ -elimination sequence is a sequence of $\varepsilon\tau$ -terms e_i and sets of critical formulas $\Lambda(e_i)$ belonging to it, such that eliminating $\Lambda_i(e_i)$ from proof π_i results in a new proof π_{i+1} of a disjunction of instances of $D(e_i)$. This new proof proceeds from instances of the critical formulas for which e_i is not critical and the remaining critical formulas belonging to e_i . Since the definition requires π_i to be an $\varepsilon\tau$ -proof, the formulas in $\Gamma[T_i]$ must actually be critical formulas.

If an $\varepsilon\tau$ -elimination sequence exists for a formula E and its $\varepsilon\tau$ -proof π_0 , then the extended first $\varepsilon\tau$ -theorem holds for E :

PROPOSITION 9.2. *Suppose π is an $\mathbf{L}\varepsilon\tau$ proof of $E(u_1, \dots, u_n)$ where $E(x_1, \dots, x_n)$ is $\varepsilon\tau$ -free. Suppose furthermore that an $\varepsilon\tau$ -elimination sequence exists for π . Then there are tuples of terms t_{i1}, \dots, t_{in} such that $\mathbf{L} \vdash \bigvee_{i=1}^l E(t_{i1}, \dots, t_{in})$.*

PROOF. Since $E(x_1, \dots, x_n)$ is $\varepsilon\tau$ -free, $E(t_{i1}, \dots, t_{in})[s/e] \equiv E(t_{i1}[s/e], \dots, t_{in}[s/e])$. The result follows by induction on k , the length of the $\varepsilon\tau$ -elimination sequence for π . ◻

For the proof of the extended first $\varepsilon\tau$ -theorem, then, it is sufficient to show that suitable e -elimination sets always exist, and that $\varepsilon\tau$ -terms e and sets of associated critical formulas can be successively chosen in such a way as to yield an $\varepsilon\tau$ -elimination sequence for π_1 . Hilbert and Bernays did this by defining a well-ordering of $\varepsilon\tau$ -terms with the property that eliminating maximal $\varepsilon\tau$ -terms according to this ordering guarantees that the $\Gamma[T_i]$ are again critical formulas, and that in each step no critical $\varepsilon\tau$ -terms are newly introduced which are larger (in the ordering).

The ordering used is the lexicographic order on two complexity measures of $\varepsilon\tau$ -terms e . We say that e is *nested in* an $\varepsilon\tau$ -term e' if e is a proper subterm of e' , i.e., if every occurrence of a variable which is free in e is also free in e' . An $\varepsilon\tau$ -term e is *subordinate* to $e' \equiv \varepsilon_x A(x)$ or $\equiv \tau_x A(x)$ if e occurs in e' and x is free in e . The *degree* $\text{deg}(e)$ of e is the maximal level of nesting of subterms of e ; the rank $\text{rk}(e)$ the maximal level of subordination. When C is a critical formula belonging to e , we let $\text{deg}(C) = \text{deg}(e)$ and $\text{rk}(C) = \text{rk}(e)$.

The Hilbert-Bernays elimination order proceeds by always picking a critical formula of maximal degree among the critical formulas of maximal rank. Its success relies on the following two lemmas, which establish that (a) replacement of maximal $\varepsilon\tau$ -terms in a critical formula results in a critical formula, and (b) if new critical formulas are generated, they are of lower rank, or of the same rank but of lower degree.

LEMMA 9.3. *Suppose e is an $\varepsilon\tau$ -term, t any term, C is a critical formula for which e is not critical, and $\text{rk}(C) \leq \text{rk}(e)$. Then $C[t/e]$ is also a critical formula.*

LEMMA 9.4. *Suppose e is an $\varepsilon\tau$ -term, t any term, C is a critical formula for which e is not critical, $\text{rk}(C) \leq \text{rk}(e)$, and if $\text{rk}(C) = \text{rk}(e)$ then $\text{deg}(C) \leq \text{deg}(e)$. Then $\text{rk}(C[t/e]) \leq \text{rk}(e)$, and if $\text{rk}(C[t/e]) = \text{rk}(e)$, then $\text{deg}(C[t/e]) \leq \text{deg}(e)$.*

The proofs of the preceding lemmas are as in Hilbert and Bernays [1939]; see also Moser and Zach [2006, §5] and Zach [2017, §4.1].

PROPOSITION 9.5. *Suppose π is an $\mathbf{L}\varepsilon\tau$ proof of $E(u_1, \dots, u_n)$ where $E(x_1, \dots, x_n)$ is $\varepsilon\tau$ -free. Suppose furthermore that for every $\mathbf{L}\varepsilon\tau$ proof and critical $\varepsilon\tau$ -term e there is a complete e -elimination set. Then there is a complete $\varepsilon\tau$ -elimination sequence for π .*

PROOF. Take $\pi_0 = \pi$. Suppose π_i has been defined. Let e_i be a critical $\varepsilon\tau$ -term of π_i of maximal degree among the critical terms of maximal rank. Let $\Lambda_i(e_i)$ be all critical formulas belonging to e_i , and Γ_i the remaining critical formulas. We have

$$\Gamma_i, \Lambda_i(e_i) \vdash_{\mathbf{L}\varepsilon\tau} D_i$$

By assumption, there is a complete e -elimination set T_i for $\Lambda_i(e_i)$ and so we have π_{i+1} showing that

$$\Gamma_i[T_i] \vdash_{\mathbf{L}\varepsilon\tau} \bigvee_{t \in T_i} D_i[t/e_i]$$

Each critical formula C in Γ_i is not of higher rank than e_i , and if it is of equal rank it is not of higher degree. So by Lemma 9.3, $C[t/e_i]$ is a critical formula. Hence π_{i+1} is a correct $\mathbf{L}\varepsilon\tau$ -proof. Let $\Gamma_{i+1} = \Gamma_i[T_i]$ and $D_{i+1} \equiv \bigvee_{t \in T_i} D_i[t/e_i]$.

Eventually, $\Gamma_i = \emptyset$, since in each step, by Lemma 9.4, the maximal rank of critical $\varepsilon\tau$ -terms in Γ_i does not increase, the maximal degree of critical $\varepsilon\tau$ -terms of maximal rank does not increase, and the number of critical $\varepsilon\tau$ -terms of maximal degree among those of maximal rank decreases. \dashv

COROLLARY 9.6. *The extended first $\varepsilon\tau$ -theorem holds for \mathbf{C} .*

PROOF. By Proposition 7.7, every critical $\varepsilon\tau$ -term has complete elimination sets. So by Proposition 9.5, there always is an elimination sequence. The extended first $\varepsilon\tau$ -theorem follows by Proposition 9.2. \dashv

Remark 9.7. The traditional procedure following the Hilbert-Bernays order, which eliminates all critical formulas belonging to a maximal $\varepsilon\tau$ -term together, is not the only possible procedure that guarantees termination. We pointed out in Remark 7.8 that using Proposition 7.6 for all critical formulas belonging to a single $\varepsilon\tau$ -term results in a larger disjunction than Proposition 7.7. Despite this, the ability in the classical case to eliminate single critical formulas provides flexibility that can be exploited to produce smaller overall Herbrand disjunctions. As Baaz, Leitsch, and Lolic [2018, Theorem 3] show, there are sequences of $\varepsilon\tau$ -proofs where the original procedure produces Herbrand disjunctions that are non-elementarily larger than a more efficient elimination order.

THEOREM 9.8. *The extended first $\varepsilon\tau$ -theorem holds for negated formulas in any $\mathbf{L}\varepsilon\tau$ such that $\mathbf{L} \vdash J$, e.g., $\mathbf{KC}\varepsilon\tau$ and $\mathbf{LC}\varepsilon\tau$.*

PROOF. \mathbf{L} has complete e -elimination sets for end-formulas that are disjunctions of negated formulas, by Proposition 7.9. Note that J , i.e., $\neg A \vee \neg\neg A$, follows intuitionistically from $(A \rightarrow \neg A) \vee (\neg A \rightarrow A)$, which is an instance of *Lin*. So $\mathbf{LC} \vdash J$. \dashv

THEOREM 9.9. *The extended first $\varepsilon\tau$ -theorem holds for $\mathbf{LC}_m\varepsilon\tau$.*

PROOF. By Theorem 8.6, every critical $\varepsilon\tau$ -term has complete elimination sets. The extended first $\varepsilon\tau$ -theorem follows by Propositions 9.5 and 9.2. \dashv

THEOREM 9.10. *The first $\varepsilon\tau$ -theorem holds in $\mathbf{LC}\varepsilon\tau$, i.e., if $\mathbf{LC}\varepsilon\tau \vdash D$ for an $\varepsilon\tau$ -free formula D , then $\mathbf{LC} \vdash D$.*

PROOF. If $\mathbf{LC}\varepsilon\tau \vdash D$ then also $\mathbf{LC}_m\varepsilon\tau \vdash D$. Since D is $\varepsilon\tau$ -free, $\mathbf{LC}_m \vdash D$ by Theorem 9.9. In general $\mathbf{LC} \vdash D$ iff $\mathbf{LC}_m \vdash D$ for all m , so the claim follows. \dashv

We are indebted to the referee for the JOURNAL for the following observation:

PROPOSITION 9.11. *The first $\varepsilon\tau$ -theorem holds for negated formulas in $\mathbf{L}\varepsilon\tau$ for any intermediate logic \mathbf{L} , including \mathbf{H} : if $\mathbf{L}\varepsilon\tau \vdash \neg D$ for an $\varepsilon\tau$ -free formula D , then $\mathbf{L} \vdash D$.*

PROOF. If $\mathbf{L}\varepsilon\tau \vdash \neg D$ then also $\mathbf{C}\varepsilon\tau \vdash \neg D$. By the first ε -theorem for \mathbf{C} (Corollary 9.6), $\mathbf{C} \vdash \neg D$, and by Glivenko's Theorem, $\mathbf{H} \vdash \neg D$. \dashv

§10. Herbrand's Theorem and the Second Epsilon-Theorem. The extended first $\varepsilon\tau$ -theorem implies Herbrand's theorem for purely existential formulas. If $E \equiv \exists x_1 \dots \exists x_n E'(x_1, \dots, x_n)$ is provable in predicate logic, then so is $E^{\varepsilon\tau} \equiv E'(e_1, \dots, e_n)$ for some ε -terms e_1, \dots, e_n . From the extended first ε -theorem we then obtain a proof in propositional logic of a Herbrand disjunction

$$E'(t_1^1, \dots, t_n^1) \vee \dots \vee A(t_1^k, \dots, t_n^k)$$

for some terms t_i^j . In predicate logic, we may now successively introduce existential quantifiers to obtain the original formula E . This holds in any intermediate predicate logic in which the first $\varepsilon\tau$ -theorem holds, since the only principles used

in the last step (proving E from its Herbrand disjunction) are $A(t) \rightarrow \exists x A(x)$ and $\exists x(A(x) \vee B) \rightarrow (\exists x A(x) \vee B)$, (x not free in B) which already hold in intuitionistic logic. (Despite the failure of the extended first $\varepsilon\tau$ -theorem in $\mathbf{LC}\varepsilon\tau$, the Herbrand theorem for existential formulas does hold in \mathbf{QLC} ; see Aschieri [2017]).

For classical predicate logic, the Herbrand theorem for existential formulas implies the Herbrand theorem for prenex formulas. If a prenex formula E has a proof in first-order logic, so does its (purely existential) *Herbrand form* $H(E) = \exists x_1 \dots \exists x_n E'(x_1, \dots, x_n)$. In classical predicate logic, we can obtain not just $H(E)$ from its Herbrand disjunction, but also the original prenex formula E . This requires that we introduce not just existential quantifiers, but also universal quantifiers. Consequently, we need not just the generalization rule $A(x)/\forall x A(x)$ but also a principle that allows us to shift universal quantifiers over disjunctions, viz.,

$$\forall x(A(x) \vee B) \rightarrow (\forall x A(x) \vee B). \quad (CD)$$

This principle is not intuitionistically valid: it characterizes Kripke frames with constant domains. Since the implication $E \rightarrow H(E)$ already holds in intuitionistic logic, any intermediate predicate logic in which CD holds and which has the extended first ε -theorem also has Herbrand's theorem for prenex formulas:

PROPOSITION 10.1. *Suppose $\mathbf{QL} + Ax$ proves CD and $\mathbf{L}\varepsilon\tau$ has the extended first $\varepsilon\tau$ -theorem. Then $\mathbf{QL} + Ax$ has Herbrand's theorem for prenex formulas.*

Although the extended first ε -theorem implies Herbrand's theorem for prenex formulas if CD is provable, the converse is not true. As we showed in Theorem 6.2, the extended first $\varepsilon\tau$ -theorem holds for $\mathbf{L}\varepsilon\tau$ only if $\mathbf{L} \vdash B_m$ for some m . In particular, it does not hold for infinite-valued first-order Gödel logics $\mathbf{G}_{\mathbb{R}} = \mathbf{QLC} + CD$. However, Herbrand's theorem for prenex formulas does hold for $\mathbf{G}_{\mathbb{R}}$; see Baaz, Preining, and Zach [2003, Theorem 5.7] and Baaz, Preining, and Zach [2007, Theorem 7.8]. (Incidentally, the Herbrand theorem for prenex formulas also holds in intuitionistic logic despite the invalidity of CD ; see Bowen [1976]).

In classical logic, the second ε -theorem can be proved using the extended first ε -theorem as follows: Suppose $\mathbf{C}\varepsilon\tau \vdash A^{\varepsilon\tau}$. Since A is equivalent to a prenex formula A^p in classical logic, we have $\mathbf{QC} \vdash A \rightarrow A^p$. Prenex formulas imply their Herbrand forms, i.e., $\mathbf{QC} \vdash A^p \rightarrow H(A^p)$. Together we have $\mathbf{QC} \vdash A \rightarrow H(A^p)$ and by translating into the $\varepsilon\tau$ -calculus, $\mathbf{C}\varepsilon\tau \vdash A^{\varepsilon\tau} \rightarrow H(A^p)^{\varepsilon\tau}$, so $\mathbf{C}\varepsilon\tau \vdash H(A^p)^{\varepsilon\tau}$. By the extended first $\varepsilon\tau$ -theorem, $H(A^p)$ has a Herbrand disjunction, from which (in \mathbf{QC}) we can prove A^p and hence A .

The steps that may fail in an intermediate predicate logic $\mathbf{QL} + Ax$, other than the extended first $\varepsilon\tau$ -theorem, are the provability of $A \leftrightarrow A^p$ and proving A^p from the Herbrand disjunction of $H(A^p)$. These steps do work provided all quantifier shifts can be carried out (i.e., in addition to CD also the formulas Q_{\forall} and Q_{\exists}). Thus, if $\mathbf{L}\varepsilon\tau$ has the extended first $\varepsilon\tau$ -theorem, any intermediate predicate logic $\mathbf{QL} + Ax$ in which all quantifier shifts are provable also has the second $\varepsilon\tau$ -theorem:

PROPOSITION 10.2. *Suppose $\mathbf{QL} + Ax$ proves CD , Q_{\exists} , and Q_{\forall} , and $\mathbf{LE}\tau$ has the extended first $\varepsilon\tau$ -theorem. Then $\mathbf{QL} + Ax$ has the second $\varepsilon\tau$ -theorem, i.e., if $\mathbf{LE}\tau \vdash A^{\varepsilon\tau}$ then $\mathbf{QL} + Ax \vdash A$.*

PROOF. Suppose $\mathbf{LE}\tau \vdash A^{\varepsilon\tau}$. The $\varepsilon\tau$ -calculus proves $\varepsilon\tau$ -translations of all quantifier shifts, so $\mathbf{LE}\tau \vdash (A^p)^{\varepsilon\tau}$, and since $\mathbf{QH} \vdash A^p \rightarrow H(A^p)$ also $\mathbf{LE}\tau \vdash (H(A^p))^{\varepsilon\tau}$. By the extended first $\varepsilon\tau$ -theorem for $\mathbf{LE}\tau$, there is a Herbrand disjunction A' of $H(A^p)$ so that $\mathbf{L} \vdash A'$. Since $\mathbf{QL} + Ax \vdash CD$, $\mathbf{QL} + Ax \vdash A^p$. Since $\mathbf{QL} + Ax \vdash Q_{\exists}$ and $\mathbf{QL} + Ax \vdash Q_{\forall}$, also $\mathbf{QL} + Ax \vdash A^p \rightarrow A$, and so $\mathbf{QL} + Ax \vdash A$. \dashv

Infinite-valued first-order Gödel logic $\mathbf{G}_{\mathbb{R}} = \mathbf{QLC} + CD$ is an intermediate pre- $\varepsilon\tau$ -logic which proves CD but not Q_{\forall} or Q_{\exists} , and the extended first $\varepsilon\tau$ -theorem does not hold for $\mathbf{LC}\varepsilon\tau$. The logics of linear Kripke frames with m worlds (and varying domains) are \mathbf{QLC}_m . But \mathbf{QLC}_m does not prove CD , so it also does not have the second $\varepsilon\tau$ -theorem:

PROPOSITION 10.3. *$\mathbf{G}_{\mathbb{R}} = \mathbf{QLC} + CD$ and \mathbf{QLC}_m do not have the second $\varepsilon\tau$ -theorem.*

PROOF. We have $\mathbf{LC}\varepsilon\tau \vdash Q_{\forall}$ but $\mathbf{QLC} + CD \not\vdash Q_{\forall}$, and $\mathbf{LC}_m\varepsilon\tau \vdash CD^{\varepsilon\tau}$ but $\mathbf{QLC}_m \not\vdash CD$. \dashv

However, we have:

COROLLARY 10.4. *The second $\varepsilon\tau$ -theorem holds for finite-valued first-order Gödel logics $\mathbf{G}_m = \mathbf{QLC}_m + CD$.*

PROOF. By Proposition 10.2, as \mathbf{G}_m proves CD , Q_{\exists} , and Q_{\forall} and has the extended first $\varepsilon\tau$ -theorem. \dashv

Herbrand's theorem also yield other results, for instance the following.

PROPOSITION 10.5. *Suppose $\mathbf{QL} + Ax_1 \subseteq \mathbf{QL} + Ax_2$ are intermediate predicate logics. If Herbrand's theorem holds in $\mathbf{QL} + Ax_2$ for existential formulas A , then $\mathbf{QL} + Ax_1 \vdash A$ iff $\mathbf{QL} + Ax_2 \vdash A$. The result also holds for prenex formulas A if $\mathbf{QL} + Ax_1 \vdash CD$.*

PROOF. The "only if" direction is trivial since $\mathbf{QL} + Ax_1 \subseteq \mathbf{QL} + Ax_2$. For the "if" direction, assume $\mathbf{QL} + Ax_2 \vdash A$. Then there is a Herbrand disjunction A' of A provable in \mathbf{L} and hence in $\mathbf{QL} + Ax_1$. Since $\mathbf{QH} \vdash A' \rightarrow A$, also $\mathbf{QL} + Ax_1 \vdash A$.

Now suppose in addition that $\mathbf{QL} + Ax_1 \vdash CD$ and $\mathbf{QL} + Ax_2 \vdash A$ with A prenex. Then since $\mathbf{QH} \vdash A \rightarrow H(A)$, $\mathbf{QL} + Ax_2 \vdash H(A)$. By Herbrand's theorem there is a Herbrand disjunction A' and $\mathbf{L} \vdash A'$. From A' , $\mathbf{QL} + Ax_1$ proves A , using just intuitionistically valid inferences as well as CD . \dashv

COROLLARY 10.6. *The existential fragments of \mathbf{QLC} and $\mathbf{G}_{\mathbb{R}}$ agree.*

PROOF. Take $\mathbf{L} = \mathbf{LC}$, $Ax_1 = \emptyset$, and $Ax_2 = CD$. Infinite-valued Gödel logic $\mathbf{G}_{\mathbb{R}} = \mathbf{QLC} + CD$ has Herbrand's theorem. \dashv

Whenever the extended first $\varepsilon\tau$ -theorem holds for $\mathbf{L}\varepsilon\tau$, Herbrand's theorem for existential formulas also holds for any intermediate predicate logic containing \mathbf{L} . So, whenever an intermediate predicate logic $\mathbf{QL} + Ax_1 \subseteq \mathbf{QL} + Ax_2$, and $\mathbf{L}\varepsilon\tau$ has the extended first $\varepsilon\tau$ -theorem, the purely existential fragments of $\mathbf{QL} + Ax_1$ and $\mathbf{QL} + Ax_2$ agree. For instance:

COROLLARY 10.7. *The existential fragments of \mathbf{QLC}_m and $\mathbf{G}_m = \mathbf{QLC}_m + CD$ agree.*

By similar reasoning, the result holds for formulas of the form $\exists \vec{x} \neg A(\vec{x})$ for \mathbf{KC} and its extensions, since \mathbf{KC} has the extended first $\varepsilon\tau$ -theorem for negated formulas (Theorem 9.8). By Proposition 6.7, the result can also be extended to formulas of the form $\forall \vec{y} B(\vec{y}) \rightarrow \exists \vec{x} A(\vec{x})$.

§11. Elimination of Critical Formulas using *Lin*. In the classical case, the first ε -theorem is obtained by successively eliminating critical formulas belonging to a single ε -term using excluded middle. In intermediate logics this is not available, but as we have seen in the proof of Theorem 8.6, critical formulas can also be eliminated using B_m and *Lin*. And in fact, if a critical $\varepsilon\tau$ -term e has only predicative critical formulas then it has a complete e -elimination set (by Lemma 8.5) already in \mathbf{LC} , since only *Lin* is required to eliminate predicative critical formulas. Thus, the procedure of the first $\varepsilon\tau$ -theorem terminates for all $\varepsilon\tau$ -proofs in which no impredicative critical formulas occurs during the successive elimination of critical formulas.

PROPOSITION 11.1. *If $\mathbf{LC}\varepsilon\tau \vdash E(u_1, \dots, u_n)$ and there is an elimination sequence $\langle \pi_i, \Lambda_i(e_i), T_i \rangle$ in which each formula in $\Lambda_i(e_i)$ is predicative, then $\mathbf{LC} \vdash \bigvee E(t_{i1}, \dots, t_{in})$.*

PROOF. By Proposition 9.2, since if in each step of the elimination sequence the eliminated critical formulas are all predicative, the elimination already works in \mathbf{LC} by Lemma 8.5. \dashv

So if there is a way to select critical $\varepsilon\tau$ -terms e_i successively for elimination in such way that the critical formulas belonging to e_i are always predicative, the extended first $\varepsilon\tau$ -theorem holds in $\mathbf{LC}\varepsilon\tau$ for a particular proof π . However, it is hard to determine just by inspecting π if this is possible. For one, it is not sufficient that the critical formulas in π itself are all predicative: eliminating the critical formulas belonging to one critical $\varepsilon\tau$ -term may turn a remaining predicative critical formula into an impredicative one. For instance, consider

$$\begin{aligned} E(x, y) &\equiv (A(f(y)) \rightarrow A(x)) \wedge (B(g(x)) \rightarrow B(y)) \\ e_A &\equiv \varepsilon_x A(x) \\ e_B &\equiv \varepsilon_y B(y) \end{aligned}$$

Then $D(e_A, e_B)$ has an $\mathbf{LC}\varepsilon\tau$ proof, since it is the conjunction of the critical formulas

$$\begin{aligned} A(f(e_B)) &\rightarrow A(e_A) \\ B(g(e_A)) &\rightarrow B(e_B) \end{aligned}$$

which are both predicative. If we first eliminate e_A we would replace e_A by $f(e_B)$ in the second, resulting in

$$B(g(f(e_B))) \rightarrow B(e_B)$$

which is impredicative. Similarly, eliminating e_B leaves the impredicative

$$A(f(g(e_A))) \rightarrow A(e_A).$$

So no elimination sequence resulting in only predicative critical formulas at every step is possible.

Of course, if the term t in a critical formula $A(t) \rightarrow A(e)$ contains no $\varepsilon\tau$ -term at all, it is predicative, and replacing some $\varepsilon\tau$ -term e' in it by a term t' cannot result in an impredicative critical formula. Let us call such critical formulas *weak*.

DEFINITION 11.2. A critical formula $A(t) \rightarrow A(\varepsilon_x A(x))$ resp. $A(\tau_x A(x)) \rightarrow A(t)$ is *weak* in π if t does not contain any critical ε - or τ -term of π .

If the critical formulas in π are all weak, there is an elimination sequence.

PROPOSITION 11.3. *Suppose $\mathbf{LC}\varepsilon\tau \vdash E(e_1, \dots, e_n)$ with a proof in which all critical formulas are weak. Then there are terms t_i^j such that $\mathbf{LC} \vdash \bigvee_j E(t_1^j, \dots, t_n^j)$.*

PROOF. Take a critical $\varepsilon\tau$ -term e of maximum degree among those of maximal rank in π , let $\Gamma(e)$ be the critical formulas belonging to e , Γ the remaining critical formulas, and suppose the end-formula is $D(e)$. Since all critical formulas $A(u) \rightarrow A(e)$ (or $A(e) \rightarrow A(u)$) are weak, e does not occur in u , i.e., all critical formulas in $\Gamma(e)$ are predicative. By Lemma 8.5, e has an e -elimination set T , and correspondingly $\Gamma[T] \vdash_{\mathbf{LC}} \bigvee_t D(t)$. However, since the critical formulas in Γ are also weak, they do not contain e , hence $\Gamma[T] = \Gamma$. The result follows by the same inductive proof as the first $\varepsilon\tau$ -theorem. \dashv

The extended first $\varepsilon\tau$ -theorem guarantees the existence of Herbrand disjunctions for existential theorems, i.e., if $E \equiv \exists x_1 \dots x_n D(x_1, \dots, x_n)$ and $\vdash E$ then $\vdash \bigvee_i D(t_{1i}, \dots, t_{ni})$. The existence of a Herbrand disjunction, conversely, guarantees the existence of a proof of $E^{\varepsilon\tau}$ for which a predicative elimination sequence exists.

PROPOSITION 11.4. *If $\vdash \bigvee_i D(t_{1i}, \dots, t_{ni})$ then there is an $\varepsilon\tau$ -derivation of $[\exists x_1 \dots \exists x_n D(x_1, \dots, x_n)]^{\varepsilon\tau}$ for which a predicative elimination sequence exists.*

PROOF. We give an example only. Suppose $\mathbf{LC} \vdash D(s_1, t_1) \vee D(s_2, t_2)$. First, consider

$$e(x) \equiv \varepsilon_y D(x, y)$$

Then both

$$\begin{aligned} C_1(e(s_1)) &\equiv D(s_1, t_1) \rightarrow D(s_1, e(s_1)) \\ C_2(e(s_2)) &\equiv D(s_2, t_1) \rightarrow D(s_2, e(s_2)) \end{aligned}$$

are predicative critical formulas. Since $\mathbf{LC} \vdash D(s_1, t_1) \vee D(s_2, t_2)$ we get from them $D(s_1, e(s_1)) \vee D(s_2, e(s_2))$. Now let

$$\begin{aligned} e' &\equiv \varepsilon_x D(x, e(x)). \\ C_3(e') &\equiv D(s_1, e(s_1)) \rightarrow D(e', e(e')) \\ C_4(e') &\equiv D(s_2, e(s_2)) \rightarrow D(e', e(e')) \end{aligned}$$

are also predicative critical formulas. Together we have a proof of $D(e', e(e')) \equiv [\exists x \exists y D(x, y)]^{\varepsilon\tau}$. Then

$$\begin{aligned} &\langle \{C_3(e'), C_4(e')\}, \{s_1, s_2\} \rangle \\ &\langle \{C_1(e(s_1))\}, \{t_1\} \rangle \\ &\langle \{C_2(e(s_2))\}, \{t_2\} \rangle \end{aligned}$$

is an elimination sequence. In fact it is an elimination sequence following Hilbert's ordering, since e' has higher rank than $e(s_1)$ and $e(s_2)$. In each step, only predicative critical formulas are generated. It produces the original Herbrand disjunction. \dashv

§12. The First $\varepsilon\tau$ -Theorem and Order Induction. In arithmetic, the usual methods for eliminating critical formulas based on the extended first ε -theorem do not work; and so consistency proofs for systems based on the ε -calculus use other methods such as the ε -substitution method (see Ackermann [1940] and Moser [2006]; the history of the two approaches is discussed in Zach [2004]). The methods developed for the extended first $\varepsilon\tau$ -theorem to eliminate predicative critical formulas from proofs in $\mathbf{LC}\varepsilon\tau$ above can, however, also be applied in classical theories of order (including arithmetic). Suppose T is a universal theory involving a relation $<$, and consider the order induction rule $IR_{<}$,

$$\frac{\forall x(x < y \rightarrow A(x)) \rightarrow A(y)}{A(t)} IR_{<}$$

We denote by $\vdash_{<}$ the derivability relation generated by classical logic extended by $IR_{<}$. The resulting system is equivalent to adding to classical first-order logic the order induction principle for $<$,

$$\forall y((\forall x(x < y \rightarrow A(x)) \rightarrow A(y)) \rightarrow \forall z A(z)). \quad (IP_{<})$$

PROPOSITION 12.1. $T \vdash_{<} A$ iff $T + IP_{<} \vdash A$

PROOF. The “only if” direction follows by observing that if

$$\begin{aligned} T \vdash \forall x(x < y \rightarrow A(x)) \rightarrow A(y) \text{ then also} \\ T \vdash \forall y(\forall x(x < y \rightarrow A(x)) \rightarrow A(y)) \end{aligned}$$

and so $A(t)$ follows from $IP_{<}$ and $\forall x A(x) \rightarrow A(t)$ by modus ponens. For the “if” direction, let P_A be

$$\forall y(\forall x(x < y \rightarrow A(x)) \rightarrow A(y))$$

Then by logic,

$$\begin{aligned} &\vdash \forall u(u < v \rightarrow A(u)) \rightarrow (P_A \rightarrow A(v)) \text{ and so} \\ &\vdash \forall u(u < v \rightarrow (P_A \rightarrow A(u))) \rightarrow (P_A \rightarrow A(v)) \\ &\vdash_{<} P_A \rightarrow A(z) \text{ by } IR_{<}, \text{ and consequently} \\ &\vdash_{<} P_A \rightarrow \forall z A(z). \end{aligned}$$

Thus, $\vdash_{<} IP_{<}$. ⊣

Now consider the classical ε -calculus extended by critical formulas of the form

$$A(t) \rightarrow \neg t < \varepsilon_x A(x)$$

These critical formulas are obviously equivalent to Hilbert’s “critical formulas of the second form,” $A(t) \rightarrow \varepsilon_x A(x) \leq t$, over a theory that proves that $<$ is trichotomous. If A is derivable from T and ordinary critical formulas for ε -terms and critical formulas of this second kind, we write $T \vdash_{< \varepsilon} A$. The standard translation A^ε of a formula A is defined as in Definition 3.3, except $\forall x A(x)^\varepsilon \equiv A^\varepsilon(\varepsilon_x \neg A^\varepsilon(x))$. Then we can show:

PROPOSITION 12.2. *If $T \vdash_{<} A$ then $T^\varepsilon \vdash_{< \varepsilon} A^\varepsilon$.*

PROOF. As in the proof of Lemma 3.5. We just have to deal with application of $IR_{<}$. Suppose we have a derivation of the ε -translation of the premise of $IR_{<}$,

$$(e(y) < y \rightarrow A^\varepsilon(e(y)) \rightarrow A^\varepsilon(y))$$

where $e(y) \equiv \varepsilon_x \neg(x < y \rightarrow A^\varepsilon(x))$ is the ε -term used in the translation of $\forall x(x < y \rightarrow A(x))$. By substituting $e' \equiv \varepsilon_z \neg A^\varepsilon(z)$ for y throughout the proof, we obtain

$$(e(e') < e' \rightarrow A^\varepsilon(e(e')) \rightarrow A^\varepsilon(e'))$$

Take the critical formula of second kind $\neg A^\varepsilon(t) \rightarrow \neg t < \varepsilon_z \neg A^\varepsilon(z)$ and let t be $e(e')$. By contraposition, we have

$$\begin{aligned} e(e') < e' \rightarrow A^\varepsilon(e(e')) \text{ and so} \\ A^\varepsilon(e') \end{aligned}$$

by modus ponens. The conclusion of $IR_{<}$, $A^\varepsilon(t)$, now follows from an ordinary critical formula belonging to $\neg A^\varepsilon(z)$, viz.,

$$\neg A^\varepsilon(t) \rightarrow \neg A^\varepsilon(e').$$

⊖

Arithmetic does not have a Herbrand theorem, and thus also no first ε -theorem. However, Herbrand disjunctions exists for formulas $\exists \vec{x} E(\vec{x})$ iff the critical formulas belonging to the ε -terms e_1, \dots, e_n in the standard ε -translation $E^\varepsilon(e_1, \dots, e_n)$ of $\exists x_1 \dots \exists x_n E(x_1, \dots, x_n)$ can be eliminated by a predicative elimination sequence. This mirrors the situation in **LC** discussed in Section 11. The proof that critical formulas can be eliminated if a predicative elimination sequence exists is similar. Corresponding to Lemma 8.4 we'll need the following:

LEMMA 12.3. *Let V be a finite set of variables and assume that*

$$T \vdash \forall x \neg x < x \quad (Irr)$$

$$T \vdash \forall x \forall y \forall z ((x < y \rightarrow (y < z \rightarrow x < z)) \quad (Trans)$$

Then $T \vdash \bigvee_{x \in V} \bigwedge_{y \in V} \neg y < x$.

PROOF. Suppose not. Then $T + \bigwedge_{x \in V} \bigvee_{y \in V} y < x$ would be satisfiable. Let \mathfrak{M} and s be the corresponding structure and variable assignment. Fix $x_1 \in V$. Since $\mathfrak{M}, s \models \bigvee_{y \in V} y < x_1$, for some $x_2 \in V$, $s(x_2) <^{\mathfrak{M}} s(x_1)$. Continuing, we obtain $x_1, \dots, x_n \in V$ such that $s(x_{i+1}) <^{\mathfrak{M}} s(x_i)$ for any n . Since V is finite, eventually $x_i \equiv x_{i+k}$, contradicting the assumption that any model of T makes $<$ irreflexive and transitive. ⊖

THEOREM 12.4. *Suppose T is as in Lemma 12.3. Then $T \vdash \bigvee_{i=1}^k E(t_{1i}, \dots, t_{ni})$ for some terms t_{ji} iff there is a derivation of $E^\varepsilon(e_1, \dots, e_n)$ which has a predicative elimination sequence.*

PROOF. For the “only if” part, proceed as in the procedure outlined in the proof of Proposition 11.4. For the “if” part, we have to show that if the critical formulas belonging to an ε -term are predicative, they can be eliminated. Without loss of generality we may assume that for each term t_i , a corresponding critical formula of first and of second kind are both present. So suppose e is a critical ε -term and $\Theta, \Gamma, \Pi(e), \Pi'(e) \vdash_{<\varepsilon} D(e)$ where $\Pi(e)$ and $\Pi'(e)$ consist of, respectively,

$$\begin{array}{lll} A(t_1) \rightarrow A(e), & \dots, & A(t_m) \rightarrow A(e) \\ A(t_1) \rightarrow \neg t_1 < e, & \dots, & A(t_m) \rightarrow \neg t_m < e, \end{array}$$

and Θ consists of instances of formulas in T .

Let $W = \{t_1, \dots, t_m\}$. If $V \subseteq W$, let C_V be

$$\bigwedge_{t \in V} A(t) \wedge \bigwedge_{t \in W \setminus V} \neg A(t)$$

and let $\Pi_V(e)$ be the critical formulas with terms $t_i \in V$ and $\Pi_{W \setminus V}(e) = \Pi(e) \setminus \Pi_V(e)$, and similarly for $\Pi'_V(e)$ and $\Pi'_{T \setminus V}(e)$. Since $\neg A(t) \vdash A(t) \rightarrow B$, we have $C_V \vdash \Pi_{W \setminus V}(e)$ and $C_V \vdash \Pi'_{W \setminus V}(e)$, and so

$$\Theta, \Gamma, C_V, \Pi_V(e), \Pi'_V(e) \vdash_{<\varepsilon} D(e)$$

Since the critical formulas in $\Pi(e)$ are predicative, t_i does not contain e and so $C_V[t_i/e] = C_V$. Since $C_V \vdash A(t_i)$ for every $t_i \in V$, $C_V \vdash \Pi_V(t_i)$. So we also have

$$\Theta[t_i/e], \Gamma[t_i/e], C_V, \Pi'_V(t_i) \vdash_{<\varepsilon} D(t_i)$$

for each $t_i \in V$ and consequently

$$\Theta[V], \Gamma[V], C_V, \bigvee_{t \in V} \bigwedge_{u \in V} (A(u) \rightarrow \neg u < t) \vdash_{<\varepsilon} \bigvee_{t \in V} D(t)$$

Since this is true for every $V \subseteq W$, we get

$$\Theta[V], \Gamma[W], \bigvee_{V \subseteq W} C_V, \Xi \vdash_{<\varepsilon} \bigvee_{t \in W} D(t)$$

if we let $\Xi = \{B_V : V \subseteq W\}$ where B_V is $\bigvee_{t \in V} \bigwedge_{u \in V} (A(u) \rightarrow \neg u < t)$. Now $\bigvee_{V \subseteq W} C_V$ is itself provable in classical logic. So as the result of one elimination step, we get

$$\Theta[V], \Gamma[W], \Xi \vdash_{<\varepsilon} \bigvee_{t \in W} D(t)$$

If there is a predicative elimination sequence, we have in the end terms t_{ij} such that

$$\Theta', \Xi' \vdash_{<\varepsilon} \bigvee_{i=1}^k E(t_{1i}, \dots, t_{ni})$$

where Θ' are all the instances of formulas of T produced in the elimination, and Ξ' are all the formulas of the form B_V , possibly with epsilon terms replaced by other terms. $T \vdash A$ for each $A \in \Theta'$. By Lemma 12.3,

$$T \vdash \bigvee_{t \in V} \bigwedge_{u \in V} \neg u < t$$

and so each formula $B_V \in \Xi'$ is also provable from T . Together we have,

$$T \vdash \bigvee_{i=1}^k E(t_{1i}, \dots, t_{ni})$$

+

§13. Open Problems. We have investigated the $\varepsilon\tau$ -calculi for intermediate logics, with a focus on the extended first $\varepsilon\tau$ -theorem. We showed that the only intermediate logics with an extended first $\varepsilon\tau$ -theorem are the finite-valued Gödel logics \mathbf{LC}_m , but obtained partial results for formulas of specific form or with specific kinds of proofs for other intermediate logics.

The natural next question to investigate is the second ε -theorem, i.e., to investigate extended $\varepsilon\tau$ -calculi for intermediate predicate logics and characterize those logics for which the extended $\varepsilon\tau$ -calculus is conservative. We have shown that $\mathbf{LE}\tau$ proves all quantifier shifts, so $\mathbf{LE}\tau$ is not conservative over any \mathbf{QL} where these are not provable. Note that this question is not automatically settled by the answer to the question of which logics have the extended first $\varepsilon\tau$ -theorem. Rather, it is a question orthogonal and requires other proof systems for a proper investigation, such as sequent calculi for $\varepsilon\tau$ -terms. For instance, Aguilera and Baaz [2019, Theorem 5.4] show that the standard translation $\Gamma \Rightarrow D^\varepsilon$ of a sequent $\Gamma \Rightarrow D$ is provable in a sequent calculus \mathbf{LJ}^ε for intuitionistic logic iff $\Gamma \Rightarrow D$ is provable in a special version \mathbf{LJ}^{++} of intuitionistic sequent calculus which is globally sound but allows violation of the eigenvariable condition. \mathbf{LJ}^{++} in turn proves $\Gamma \Rightarrow D$ iff $\mathbf{LJ} + CD + Q_\forall + Q_\exists \vdash \Gamma \Rightarrow D$ (Proposition 4.4). In other words, the second ε -theorem holds for intuitionistic predicate logic with all quantifier shifts.

The methods used here are closely related to the study of the behavior of Skolem functions in intermediate logics (of which $\varepsilon\tau$ -terms are in many ways a syntactic variant), see, e.g., Iemhoff [2019]. As mentioned in the introduction, other approaches to adding ε -operators to intuitionistic logic yield systems that are conservative over the original logic. Work on Skolemization in intuitionistic logic is relevant here, and suggests that conservative ε -calculi can be obtained by introducing existence predicates. The proof-theoretic approaches in the literature would benefit also from a complementary model-theoretic study. A Kripke-style semantics for $\varepsilon\tau$ -terms, with or without existence predicate, is still lacking (but see DeVidi [1995] for a semantics based on Heyting algebras).

In Section 11 we gave sufficient conditions for when $\varepsilon\tau$ -terms can be eliminated from a proof π in $\mathbf{LC}\varepsilon\tau$. Are there better (weaker) criteria that apply to more proofs? For instance, there may be certain kinds of orderings such that if the critical $\varepsilon\tau$ -terms in π and corresponding “witness terms” can be put into such an ordering, an elimination sequence in which only predicative critical formulas exists. The same question applies for the parallel condition in arithmetical theories in Theorem 12.4.

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