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Characterization of the Axiomatizable Prenex Fragments of First-Order Gödel Logics

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Abstract

The prenex fragments of first-order infinite-valued Gödel logics are classified. It is shown that the prenex Gödel logics characterized by finite and by uncountable subsets of $[0,1]$ *are axiomatizable, and that the prenex fragments of all countably infinite Gödel logics are not axiomatizable.*

1 Introduction

Gödel logics are one of the oldest and most interesting families of many-valued logics. Introduced by Gödel in [9], they provide the first examples of intermediate logics (intermediate, that is, in strength between classical and intuitionistic logics). Dummett [7] was the first to study infinite valued Gödel logics, axiomatizing the set of tautologies over infinite truth-value sets by intuitionistic logic plus the linearity axiom $(A \to B) \vee (B \to A)$. In terms of Kripke semantics, the characteristic linearity axiom picks out those accessibility relations which are linear orders.

Gödel logics have recently received increasing attention, both in terms of foundational investigations and in terms of applications. One of the most surprising recent results is that whereas there is only one infinite-valued propositional Gödel logic, there are infinitely many different logics at the first-order and already when only "fuzzy quantifiers" are added to the language [3, 4]. In light of the fact that first-order infinite-valued Łukasiewicz logic is not axiomatizable, it is perhaps also surprising that at least one infinite-valued Gödel logic *is* r.e. [10, 14].

Our aim in the present paper is to characterize the axiomatizable first-order *prenex* Gödel logics, i.e., those truth-value sets whose first-order validities in prenex form are r.e. This is a first step toward the characterization (in terms of axiomatizability) of first-order Gödel logics in general. Our result is that there is only one axiomatizable infinite-valued first-order prenex Gödel logic; it is characterized by any closed uncountable subset of $[0,1]$. In fact, we give an axiomatization based on a version of Herbrand's theorem for such truth-value sets, and then show that any countably infinite truth-value set has a set of prenex validities which is not r.e.

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2 Syntax and semantics

First-order Gödel logics are given by truth functions for the connectives and quantifiers, and a set of truth values. We work in a standard first-order language with variables (x, y, z, ...), constants (a, b, c, ...), function symbols (f, g, ...), predicate symbols (P, Q, R, ...), the predicate constant $\bot$, connectives ($\wedge$, $\vee$, $\to$) and quantifiers ($\exists$, $\forall$). $\neg A$ is defined as $A \to \bot$. The set of well-formed formulas is denoted by $\mathrm{Frm}(L)$. The sets of truth values for the systems we consider are closed subsets of $[0,1]$ containing both 0 and 1. Interpretations are defined as usual:

Definition 2.1 Let $V \subseteq [0,1]$ be some set of truth values which contains 0 and 1 and is closed in $\mathbb{R}$. A *many-valued interpretation* $\mathfrak{I} = \langle D, \mathbf{s} \rangle$ *based on* V is given by a *domain* D and a *valuation function* $\mathbf{s}$ which maps n-ary relation symbols to functions $D^n \to V$, $\mathbf{s}(\bot) = 0$, n-ary function symbols to functions from D^n to D, and constants of $L^{\mathfrak{I}}$ and variables to elements of D. $L^{\mathfrak{I}}$ is L extended by constant symbols for all $d \in D$; if $d \in D$, then $\mathbf{s}(d) = d$.

$\mathbf{s}$ can be extended in the obvious way to a function on all terms in $L^{\mathfrak{I}}$. The valuation of formulas in $L^{\mathfrak{I}}$ is defined by:

(1) $\mathfrak{I}(P(t_1, \ldots, t_n)) = \mathbf{s}(P)(\mathbf{s}(t_1), \ldots, \mathbf{s}(t_n))$.

(2) $\mathfrak{I}(A \wedge B) = \min(\mathfrak{I}(A), \mathfrak{I}(B))$.

(3) $\mathfrak{I}(A \vee B) = \max(\mathfrak{I}(A), \mathfrak{I}(B))$.

(4) $\mathfrak{I}(A \to B) = \mathfrak{I}(B)$ if $\mathfrak{I}(A) > \mathfrak{I}(B)$ and $= 1$ otherwise.

Since we defined $\neg A \equiv A \to \bot$, we also have $\mathfrak{I}(\neg A) = 0$ if $\mathfrak{I}(A) > 0$ and $= 1$ otherwise.

The set $\mathrm{Distr}_{\mathfrak{I}} A(x) = \{\mathfrak{I}(A(d)) : d \in D\}$ is called the *distribution* of $A(x)$. The valuations of quantified formulas are defined by infimum and supremum of their distributions.

(5) $\mathfrak{I}(\forall x A(x)) = \inf \mathrm{Distr}_{\mathfrak{I}} A(x)$.

(6) $\mathfrak{I}(\exists x A(x)) = \sup \mathrm{Distr}_{\mathfrak{I}} A(x)$.

$\mathfrak{I}$ *satisfies* a formula A, $\mathfrak{I} \models A$, if $\mathfrak{I}(A) = 1$.

Definition 2.2 The *Gödel logic* $\mathbf{G}_V$ based on a set of truth values V is the set of all $A \in \mathrm{Frm}(L)$ s.t. $\mathbf{G}_V \models A$, i.e., for every $\mathfrak{I}$ based on V, $\mathfrak{I} \models A$. The logics $\mathbf{G}_\mathbb{R}$, $\mathbf{G}_\downarrow$, $\mathbf{G}_\uparrow$, $\mathbf{G}_m$ are based on the truth value sets

$$V_\mathbb{R} = [0,1] \qquad V_\downarrow = \{\tfrac{1}{k} : k \geq 1\} \cup \{0\}$$
$$V_\uparrow = \{1 - \tfrac{1}{k} : k \geq 1\} \cup \{1\} \quad V_m = \{0, \tfrac{1}{2}, \tfrac{2}{3}, \ldots, \tfrac{m-2}{m-1}, 1\}.$$

3 Relationships between Gödel logics

In the propositional case, the relationships between finite and infinite valued Gödel logics are well understood. Any choice of an infinite set of truth-values results in the same propositional Gödel logic, viz., Dummett's **LC** [7]. Furthermore, we know that **LC** is the intersection of all finite-valued propositional Gödel logics, and that it is axiomatized by intuitionistic propositional logic **IPL** plus the schema $(A \to B) \vee (B \to A)$. **IPL** is contained in all Gödel logics (finite- or infinite-valued).

In the first-order case, the relationships are somewhat more involved. First of all, let us note that Intuitionistic predicate logic **IL** is contained in all first-order Gödel logics, since the axioms and rules of **IL** are sound for the Gödel truth functions. As a consequence, we will be able to use any intuitionistically sound rules and intuitionistically valid formulas when working in any of the Gödel logics.

Proposition 3.1 *(1)* $\mathbf{G}_m \supsetneq \mathbf{G}_{m+1}$, *(2)* $\mathbf{G}_m \supsetneq \mathbf{G}_\uparrow \supsetneq \mathbf{G}_\mathbb{R}$, *(3)* $\mathbf{G}_m \supsetneq \mathbf{G}_\downarrow \supsetneq \mathbf{G}_\mathbb{R}$.

Proof. The only nontrivial part is proving that the containments are strict. For this note that $(A_1 \to A_2) \vee \ldots \vee (A_m \to A_{m+1})$ is valid in $\mathbf{G}_m$ but not in $\mathbf{G}_{m+1}$. Furthermore, let $C_\uparrow = \exists x(A(x) \to \forall y A(y))$ and $C_\downarrow = \exists x(\exists y A(y) \to A(x))$. $C_\downarrow$ is valid in all $\mathbf{G}_m$ and in $\mathbf{G}_\uparrow$ and $\mathbf{G}_\downarrow$; $C_\uparrow$ is valid in all $\mathbf{G}_m$ and in $\mathbf{G}_\uparrow$, but not in $\mathbf{G}_\downarrow$; neither is valid in $\mathbf{G}_\mathbb{R}$ [3, Corollary 2.9]. ☐

The formulas $C_\uparrow$ and $C_\downarrow$ are of some importance in the study of first-order infinite-valued Gödel logics. $C_\uparrow$ expresses the fact that every infimum in the set of truth values is a minimum, and $C_\downarrow$ states that every supremum (except possibly 1) is a maximum. The only three quantifier shifting rules which are not intuitionistically valid are:

$$\forall x(A(x) \vee B) \;\to\; (\forall x A(x) \vee B) \qquad (S_1)$$
$$(B \to \exists x A(x)) \;\to\; \exists x(B \to A(x)) \qquad (S_2)$$
$$(\forall x A(x) \to B) \;\to\; \exists x(A(x) \to B) \qquad (S_3)$$

(x is not free in B.) Of these, S_1 is valid in any Gödel logic. S_2 and S_3 imply $C_\downarrow$ and $C_\uparrow$, respectively (take $\exists y A(y)$ and $\forall y A(y)$, respectively, for B). S_2 and S_3 are, respectively, both valid in $\mathbf{G}_\uparrow$, invalid and valid in $\mathbf{G}_\downarrow$, and both invalid in $\mathbf{G}_\mathbb{R}$. $\mathbf{G}_\uparrow$ is thus the only Gödel logic where every formula is equivalent to a prenex formula. This also implies that $\mathbf{G}_\uparrow \neq \mathbf{G}_\downarrow$. In fact, we have $\mathbf{G}_\downarrow \subsetneq \mathbf{G}_\uparrow$; this follows from the following theorem.

Theorem 3.2 $\mathbf{G}_\uparrow = \bigcap_{m \geq 2} \mathbf{G}_m$

Proof. By Proposition 3.1, $\mathbf{G}_\uparrow \subseteq \bigcap_{m \geq 2} \mathbf{G}_m$. We now show the reverse inclusion. Since all quantifier shifting rules are valid in $\mathbf{G}_\uparrow$, any formula A is equivalent to a prenex formula. For any given prenex formula $A \equiv \mathsf{Q}_1 x_1 \ldots \mathsf{Q}_n x_n B(\bar{x})$ (B quantifier free) we may define the *Herbrand form* A^H of A as usual as $\exists x_{i_1} \ldots \exists x_{i_m} B(t_1, \ldots, t_n)$, where $\{x_{i_j} : 1 \leq j \leq m\}$ is the set of existentially quantified variables in A, and t_i is x_{i_j} if $i = i_j$, or is $f_i(x_{i_1}, \ldots, x_{i_k})$ if x_i is universally quantified and $k = \max\{j : i_j < i\}$. We will write $B(t_1, \ldots, t_n)$ as $B^F(x_{i_1}, \ldots, x_{i_m})$ if we want to emphasize the free variables.

Lemma 3.3 $\mathbf{G}_\uparrow \models A$ *iff* $\mathbf{G}_\uparrow \models A^H$.

Proof. If: Suppose that $\mathbf{G}_\uparrow \not\models A$. In $\mathbf{G}_\uparrow$, every infimum is a minimum, i.e., if $\inf \mathrm{Distr}_\mathfrak{I} A(x) = v$ then for some $d \in D$, $\mathfrak{I}(A(d)) = v$. Hence, we can extend $\mathfrak{I}$ by interpretations for the f_i as in the classical case. Only if: Obvious. ☐

It now suffices to show that if $\mathbf{G}_\uparrow \not\models A$, where A is existential, then $\mathbf{G}_m \not\models A$ for some m. For this we need the following

Lemma 3.4 *Given* $\mathfrak{I} = \langle D, \mathbf{s} \rangle$ *and* $v < 1$, *define* $\mathfrak{I}_v = \langle D, \mathbf{s}_v \rangle$ *where* $\mathbf{s}_v(P)(d_1, \ldots, d_n) = \mathfrak{I}(P(d_1, \ldots, d_n))$ *if* $\mathfrak{I}(P(d_1, \ldots, d_n)) \leq v$ *and* $= 1$ *otherwise. If* $A \in \mathrm{Frm}(L^\mathfrak{I})$ *does not contain* $\forall$, *then* $\mathfrak{I}_v(A) = 1$ *if* $\mathfrak{I}(A) > v$ *and* $\mathfrak{I}_v(A) = \mathfrak{I}(A)$ *if* $\mathfrak{I}(A) \leq v$.

Proof. By induction on the complexity of A. ☐

Now suppose there is an interpretation $\mathfrak{I}$ s.t. $\mathfrak{I} \not\models A$. Then clearly $\sup \mathrm{Distr}_\mathfrak{I} B(\bar{x}) = v < 1$ (where $A \equiv \exists \bar{x} B(\bar{x})$) and there are only finitely many truth values below v in $V_\uparrow$, say $v = 1 - 1/k$. Construct $\mathfrak{I}_v$ as in the previous lemma. Since $\mathfrak{I}(B(\bar{d})) \leq v < 1$, by the lemma $\mathfrak{I}_v(B(\bar{d})) \leq v$ and so $\sup \mathrm{Distr}_{\mathfrak{I}_v} B(\bar{x}) \leq v$. But $\mathfrak{I}_v$ is a $\mathbf{G}_{k+1}$ interpretation, and $\mathfrak{I}_v \not\models A$. ☐

Corollary 3.5 $\mathbf{G}_m \supsetneq \bigcap_m \mathbf{G}_m = \mathbf{G}_\uparrow \supsetneq \mathbf{G}_\downarrow \supsetneq \mathbf{G}_\mathbb{R}$

One basic but important result is that the set of validities of $\mathbf{G}_V$ only depends on the order type of V. Let V, V' be two truth value sets, X a set of atomic formulas, and suppose there is an order-preserving injection $f : \{\mathfrak{I}(B) : B \in X\} \to V'$ which is so that $f(1) = 1$ and $f(0) = 0$. (Call any such f a *truth value injection on* X.) By a simple induction on A, we have:

Proposition 3.6 *Let A be a quantifier free formula, and X its set of atomic subformulas. If $\mathfrak{I}$, $\mathfrak{I}'$ are interpretations on V, V', respectively, and f is a truth value injection on X, then* $f(\mathfrak{I}(A)) = \mathfrak{I}'(A)$.

4 Order theoretical preliminaries

We will characterize the axiomatizable Gödel logics by topological and order-theoretic properties of the underlying truth value set. The most important of these properties as regards axiomatizability is the existence of a non-trivial dense linear subordering of the truth value set, i.e., a subset

$V' \subseteq V$, $|V'| \geq 2$, where for all $u, v \in V'$ with $u < v$ there is a $w \in V'$ such that $u < w < v$. In this section we show that there is such a dense subordering iff V is uncountable.

Dense linear orderings are connected to uncountable sets via perfect sets. We begin by listing some results about perfect sets from [11]. All the following notations, lemmas, theorems are carried out within the framework of Polish spaces, i.e., separable completely metrizable topological spaces. For our discussion it is only necessary to know that any closed subset of $\mathbb{R}$ (and hence, any truth-value set) is such a Polish space.

Definition 4.1 A *limit point* of a topological space is a point that is not isolated, i.e., for every open neighbourhood U of x there is a point $y \in U$ with $y \neq x$. A space is *perfect* if all its points are limit points.

It is obvious that intervals of the real line are perfect, but there are perfect sets which are not intervals:

Example 4.2 The set of all numbers in the unit interval which can be expressed in triadic notation using only 0 and 2 is called the *Cantor set*; it is a perfect subset of $[0, 1]$.

Proposition 4.3 *For any perfect set $P \subseteq \mathbb{R}$ there is a unique partition of $\mathbb{R}$ into countably many intervals such that the intersection of P with each interval is either empty, the entire interval or isomorphic to the Cantor set.*

Proof. See [15], Proposition 1 and discussion. □

To obtain a connection between uncountable sets and perfect sets we first note that it is possible to embed the Cauchy space into any perfect space, which yields

Lemma 4.4 *If X is a nonempty perfect Polish space, then the cardinality of X is $2^{\aleph_0}$; thus all nonempty perfect subsets of $\mathbb{R}$ have cardinality of the continuum.*

Proof. [11, Corollary 6.3]. □

For the other direction, we want to partition an uncountable set into a perfect kernel and a countable rest. This is the well known Cantor-Bendixon Theorem:

Theorem 4.5 (Cantor-Bendixon) *Let X be a Polish space. Then $X = P \cup C$, with P a perfect subset of X and C countable open. P is called the* perfect kernel *of X.*

As a corollary we obtain that any uncountable Polish space contains a perfect set, and therefore has cardinality $2^{\aleph_0}$. Now we can prove the central theorem:

Theorem 4.6 *A truth value set (i.e., a closed subset of $[0, 1]$) is uncountable iff it contains a non-trivial dense linear subordering.*

Proof. If: Every countable non-trivial dense linear order has order type η, $\mathbf{1} + \eta$, $\eta + \mathbf{1}$, or $\mathbf{1} + \eta + \mathbf{1}$ [13, Corollary 2.9], where η is the order type of $\mathbb{Q}$. The completion of any ordering of order type η has order type λ, the order type of $\mathbb{R}$ [13, Theorem 2.30], thus the truth value set must be uncountable.

Only if: We define a dense linear subordering for any uncountable set. In fact we will give a dense linear subordering of the perfect kernel.

Since every perfect subset of the real line is a union of intervals and sets isomorphic to the Cantor set (Proposition 4.3), it suffices to show the claim for those sets. For intervals the claim is trivial. Now consider the border points in a Cantor set, i.e., points which can only be approximated within the Cantor set from above or below but not both. In the ternary notation these are the points with a finite number of 0 or a finite number of 2, i.e., their ternary expansions are either $a = 0.a_1a_2 \ldots a_n$ or $b = 0.b_1b_2 \ldots b_n2222\ldots$ Each border point can be approximated by a sequence of inner points a^k. For the k-th sequence element approximating a border point a we get a^k by appending $2k$ zeros and then a sequence of $020202\ldots$ at the end ($a^k = 0.a_1 \ldots a_n(00)^k\overline{02}$). For the k-th sequence element approximating a border point b we define an approximating sequence b^k by replacing the ternary expansion starting from the $2k$-th 2 with a sequence of 02's ($b^k = 0.b_1 \ldots b_n(22)^k\overline{02}$). The set of approximations of all border points is a dense subset: If $a^k = 0.a_1 \ldots a_n(00)^k\overline{02}$ and $a^{k+1} = 0.a_1 \ldots a_n(00)^k00\overline{02}$ are adjacent points in the sequence, then $a' = 0.a_1 \ldots a_n(00)^k00022222\ldots$ is a border point with $a^{k+1} < a' < a^k$, hence there are infinitely many points a'^{ℓ} between a^k and a^{k+1} in the subset. Similarly for adjacent elements of a b-sequence. The set of border points is countable, therefore the set containing all the approximation sequences is countable and has all the necessary properties. □

Note that for example 1/3 and 2/3 would not be in the dense linear subordering, because between them there is no point of the perfect set. We would replace 1/3 by a sequence of inner points approximating 1/3 from below and replace 2/3 by a sequence of inner points approximating 2/3 from above.

5 Axiomatizability results

Throughout this section, V is a truth value set which is either finite or uncountable. Let $\mathbf{G}_V$ be a Gödel logic with such a truth value set. We show how to effectively associate with each prenex formula A a quantifier-free formula A^* which is valid in $\mathbf{G}_V$ if and only if A is valid. The axiomatizability of $\mathbf{G}_V$ then follows from the axiomatizability of $\mathbf{LC}$ (in the infinite-valued case) and propositional $\mathbf{G}_m$ (in the finite-valued case). Recall that A^H stands for the Herbrand normal form of A (see the proof of Theorem 3.2).

Lemma 5.1 *If A is prenex and $\mathbf{G}_V \models A$, then $\mathbf{G}_V \models A^H$.*

Proof. Follows from the usual laws of quantification. □

Our next main result will be Herbrand's theorem for $\mathbf{G}_V$ for V uncountable. (By Theorem 4.6, V contains a dense linear subordering.) Let A be a formula. The *Herbrand universe* $U(A)$ of A is the set of all variable-free terms which can be constructed from the set of function symbols occurring in A. To prevent $U(A)$ from being finite or empty we add a constant and a function symbol of positive arity if no such symbols appear in A. The *Herbrand base* $B(A)$ is the set of atoms constructed from the predicate symbols in A and the terms of the Herbrand universe. In the next theorem we will consider the Herbrand universe of a formula $\exists\bar{x}A(\bar{x})$. We fix a non-repetitive enumeration $A_1, A_2, \ldots$ of $B(A)$, and let $X_\ell = \{\bot, A_1, \ldots, A_\ell, \top\}$ (we may take $\top$ to be a formula which is always $= 1$). $A(\bar{t})$ is an *ℓ-instance* of $A(\bar{x})$ if the atomic subformulas of $A(\bar{t})$ are in X_ℓ.

Definition 5.2 An *ℓ-constraint* is a non-strict linear ordering $\preceq$ of X_ℓ s.t. $\bot$ is minimal and $\top$ is maximal. An interpretation $\mathfrak{I}$ *fulfils* the constraint $\preceq$ provided for all $B, C \in X_\ell$, $B \preceq C$ iff $\mathfrak{I}(B) \leq \mathfrak{I}(C)$. We say that the constraint $\preceq'$ on $X_{\ell+1}$ *extends* $\preceq$ if for all $B, C \in X_\ell$, $B \preceq C$ iff $B \preceq' C$.

Proposition 5.3 *(a) Every $\mathfrak{I}$ which fulfills $\preceq'$ also fulfills $\preceq$. (b) if $\mathfrak{I}, \mathfrak{I}'$ fulfill $\preceq$, then there is a truth value injection f on X_ℓ, and $f(\mathfrak{I}(A(\bar{t}))) = \mathfrak{I}'(A(\bar{t}))$ for all ℓ-instances $A(\bar{t})$ of $A(x)$; in particular, $\mathfrak{I}(A(\bar{t})) = 1$ iff $\mathfrak{I}'(A(\bar{t})) = 1$.*

Proof. (a) Obvious. (b) Follows from Proposition 3.6. □

Lemma 5.4 *Let A be a quantifier-free formula. If $\mathbf{G}_V \models \exists\bar{x}A(\bar{x})$ then there are tuples $\bar{t}_1, \ldots \bar{t}_n$ of terms in $U(A)$, such that $\mathbf{G}_V \models \bigvee_{i=1}^{n} A(\bar{t}_i)$.*

Proof. We construct a "semantic tree" T; i.e., a systematic representation of all possible order types of interpretations of the atoms A_i in the Herbrand base. T is a rooted tree whose nodes appear at levels. Each node at level ℓ is labelled with an ℓ-constraint.

T is constructed in levels as follows: At level 0, the root of T is labelled with the constraint $\bot < \top$. Let ν be a node added at level ℓ with label $\preceq$, and let T_ℓ be the set of terms occurring in X_ℓ. Let (*) be: There is an interpretation $\mathfrak{I}$ that fulfils $\preceq$ so that for some ℓ-instance $A(\bar{t})$, $\mathfrak{I}(A(\bar{t})) = 1$. If (*) obtains, ν is a leaf node of T, and no successor nodes are added at level $\ell + 1$. Note that by Proposition 5.3, any two interpretations which fulfill $\preceq$ make the same ℓ-instances of $A(\bar{t})$ true; hence ν is a leaf node if and only if there is an ℓ-instance $A(\bar{t})$ s.t. $\mathfrak{I}(A(\bar{t})) = 1$ for *all* interpretations $\mathfrak{I}$ that fulfil $\preceq$.

If (*) does not obtain, for each $(\ell+1)$-constraint $\preceq'$ extending $\preceq$ we add a successor node ν' labelled with $\preceq'$ to ν at level $\ell + 1$.

We now have two cases:

(1) T is finite. Let $\nu_1, \ldots, \nu_m$ be the leaf nodes of T of levels $\ell_1, \ldots, \ell_m$, each labelled with a constraint $\preceq_1, \ldots, \preceq_m$. By (*), there are ℓ_i-instances $A(\bar{t}_1), \ldots, A(\bar{t}_m)$ so that $\mathfrak{I}(A(\bar{t}_i)) = 1$ for any $\mathfrak{I}$ which fulfills $\preceq_i$. It is easy to see that every interpretation fulfills at least one of the $\preceq_i$. Hence, for all $\mathfrak{I}$, $\mathfrak{I}(A(\bar{t}_1) \vee \ldots \vee A(\bar{t}_m)) = 1$, and so $\mathbf{G}_V \models \bigvee_{i=1}^{m} A(\bar{t}_i)$.

(2) T is infinite. By König's lemma, T has an infinite branch with nodes $\nu_0, \nu_1, \nu_2, \ldots$ where ν_ℓ is labelled by $\preceq_\ell$ and is of level ℓ. Each $\preceq_{\ell+1}$ extends $\preceq_\ell$, hence we can form $\preceq = \bigcup_\ell \preceq_\ell$. Let $V' \subseteq V$ be a non-trivial densely ordered subset of V, let $V' \ni c < 1$, and let $V'' = V' \cap [0, c)$. V'' is clearly also densely ordered. Now let V_c be $V'' \cup \{0, 1\}$, and let $h : B(A(x)) \cup \{\bot, \top\} \to V_c$ be an injection which is so that, for all $A_i, A_j \in B(A(x))$, $h(A_i) \leq h(A_j)$ iff $A_i \preceq A_j$, $h(\bot) = 0$ and $h(\top) = 1$. We define an interpretation $\mathfrak{I} = \langle U(A(x)), \mathbf{s} \rangle$ by: $\mathbf{s}(f)(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$ for all n-ary function symbols f and $\mathbf{s}(P)(t_1, \ldots, t_n) = h(P(t_1, \ldots, t_n))$ for all n-ary predicate symbols P (clearly then, $\mathfrak{I}(A_i) = h(A_i)$). By definition, $\mathfrak{I}$ ℓ-fulfills $\preceq_\ell$ for all ℓ. By (*), $\mathfrak{I}(A(\bar{t})) < 1$ for all ℓ-instances $A(\bar{t})$ of $A(x)$, and by the definition of V_c, $\mathfrak{I}(A(\bar{t})) < c$. Since every $A(\bar{t})$ with $\bar{t} \in U(A(x))$ is an ℓ-instance of $A(x)$ for some ℓ, we have $\mathfrak{I}(\exists x A(\bar{x})) \leq c < 1$.This contradicts the assumption that $\mathbf{G}_V \models \exists\bar{x}A(\bar{x})$. □

The following lemma establishes sufficient conditions for a logic to allow *reverse Skolemization*. By this we mean the re-introduction of quantifiers in Herbrand expansions. Here, by a *logic* $\mathbf{L}$ we mean a set of formulas that is closed under modus ponens, generalization and substitutions (of both formulas and terms). We call a formula A *valid* in $\mathbf{L}$, $\mathbf{L} \models A$, if $A \in \mathbf{L}$. The following three results follow from [1] together with Lemma 5.4:

Lemma 5.5 *Let $\mathbf{L}$ be a logic satisfying the following properties:*

(1) $\mathbf{L} \models A \vee B \Rightarrow \mathbf{L} \models B \vee A$

(2) $\mathbf{L} \models (A \vee B) \vee C \Rightarrow \mathbf{L} \models A \vee (B \vee C)$

(3) $\mathbf{L} \models A \vee B \vee B \Rightarrow \mathbf{L} \models A \vee B$

(4) $\mathbf{L} \models A(y) \Rightarrow \mathbf{L} \models \forall x A(x)$

(5) $\mathbf{L} \models A(t) \Rightarrow \mathbf{L} \models \exists x A(x)$

(6) $\mathbf{L} \models \forall x (A(x) \vee B) \Rightarrow \mathbf{L} \models \forall x A(x) \vee B$

(7) $\mathbf{L} \models \exists x (A(x) \vee B) \Rightarrow \mathbf{L} \models \exists x A(x) \vee B.$

(x is not free in B.) Let $\exists\bar{x}A^F(\bar{x})$ be the Herbrand form of the prenex formula $\overline{\mathsf{Q}}_i A(\bar{y}_i)$, and let $\bar{t}_1, \ldots, \bar{t}_m$ be tuples of terms in $U(A^F(\bar{x}))$. If $\mathbf{L} \models \bigvee_{i=1}^{m} A^F(\bar{t}_i)$, then $\mathbf{L} \models \overline{\mathsf{Q}}\bar{y}A(\bar{y})$.

Corollary 5.6 *If $\mathbf{G}_V \models \exists\bar{x}A^F(\bar{x})$, then $\mathbf{G}_V \models \overline{\mathsf{Q}}\bar{y}A(\bar{y})$.*

Theorem 5.7 *Let $A \equiv \overline{\mathsf{Q}}\bar{y}B(\bar{y})$ be prenex. $\mathbf{G}_V \models \overline{\mathsf{Q}}\bar{y}B(\bar{y})$ iff there are tuples $\bar{t}_1, \ldots \bar{t}_m$ of terms in $U(A^H(\bar{x}))$, such that $\mathbf{G}_V \models \bigvee_{i=1}^{m} B^F(\bar{t}_i)$.*

Remark 5.8 An alternative proof of Herbrand's theorem can be obtained using the analytic calculus *HIF* ("Hypersequent calculus for Intuitionistic Fuzzy logic") [6].

Theorem 5.9 *The prenex fragment of a Gödel logic based on a truth value set V which is either finite or uncountable infinite is axiomatizable. An axiomatization is given by the standard axioms and rules for* **LC** *extended by conditions (4)–(7) of Lemma 5.5 written as rules. For the m-valued case add the characteristic axiom for $\mathbf{G}_m$, $G_m \equiv \bigvee_{i=1}^{m} \bigvee_{j=i+1}^{m+1} ((A_i \to A_j) \wedge (A_j \to A_i))$.*

Proof. Completeness: Let $\overline{\mathsf{Q}}\bar{y}_i A(\bar{y})$ be a prenex formula valid in $\mathbf{G}_V$. Herbrand's theorem holds for $\mathbf{G}_V$ (for V infinite, this is Theorem 5.7; for V finite it follows from results in [2]), and so a Herbrand disjunction $\bigvee_{i=1}^{n} A^F(\bar{t}_i)$ is provable in **LC** or $\mathbf{LC} + G_m$ [8, Chapter 10.1]. $\overline{Q}\bar{y}_i A(\bar{y})$ is provable by Lemma 5.5.

Soundness: $\mathbf{G}_V$ satisfies the conditions of Lemma 5.5 (in particular, note that $\forall x(A(x) \vee B) \to (\forall x A(x) \vee B)$ with x not free in B is valid in all Gödel logics). □

6 Nonaxiomatizability results

In this section we show that the prenex fragments of first-order Gödel logics where the set of truth values does not contain a dense subset are not axiomatizable. We establish the result first for the entire set of valid formulas by reducing the classical validity of a formula in all finite models to the validity of a formula in Gödel logic (the set of these formulas is not r.e. by Trakhtenbrot's Theorem). We then strengthen the result by showing that the image of the translation from the prenex fragment of classical logic to Gödel logic is equivalent to a prenex formula.

Theorem 6.1 *If V is countably infinite, then $\mathbf{G}_V$ is not axiomatizable.*

Proof. By Theorem 4.6, V is countably infinite iff it is infinite and does not contain a non-trivial densely ordered subset. We show that for every sentence A there is a sentence A^g s.t. A^g is valid in $\mathbf{G}_V$ iff A is true in every finite (classical) first-order structure.

We define A^g as follows: Let P be a unary and L be a binary predicate symbol not occurring in A and let $Q_1, \dots, Q_n$ be all the predicate symbols in A. We use the abbreviations $x \in y \equiv \neg\neg L(x,y)$ and $x \prec y \equiv (P(y) \to P(x)) \to P(y)$. Note that for any interpretation $\mathfrak{I}$, $\mathfrak{I}(x \in y)$ is either 0 or 1, and as long as $\mathfrak{I}(P(x)) < 1$ for all x (in particular, if $\mathfrak{I}(\exists z P(z)) < 1$), we have $\mathfrak{I}(x \prec y) = 1$ iff $\mathfrak{I}(P(x)) < \mathfrak{I}(P(y))$. Let $A^g \equiv$

$$\left\{ \begin{array}{l} S \wedge c_1 \in 0 \wedge c_2 \in 0 \wedge c_2 \prec c_1 \wedge \\ \quad \forall i \big[\forall x,y \forall j \forall k \exists z K \vee \forall x \neg(x \in i)\big] \end{array} \right\} \to (A' \vee \exists u P(u))$$

where S is the conjunction of the standard axioms for 0, successor and $\leq$, with double negations in front of atomic formulas,

$$K \equiv \begin{array}{l} (j \leq i \wedge x \in j \wedge k \leq i \wedge y \in k \wedge x \prec y) \to \\ \quad \to (z \in s(i) \wedge x \prec z \wedge z \prec y) \end{array}$$

and A' is A where every atomic formula is replaced by its double negation, and all quantifiers are relativized to the predicate $R(i) \equiv \exists x(x \in i)$.

Intuitively, L is a predicate that divides a subset of the domain into levels, and $x \in i$ means that x is an element of level i. P orders the elements of the domain which fall into one of the levels in a subordering of the truth values. The idea is that for any two elements in a level $\leq i$ there is an element in level $i+1$ which lies strictly between those two elements in the ordering given by $\prec$. If this condition cannot be satisfied, the levels above i are empty. Clearly, this condition can be satisfied in an interpretation $\mathfrak{I}$ only for finitely many levels if V does not contain a dense subset, since if more than finitely many levels are non-empty, then $\bigcup_i \{\mathfrak{I}(P(d)) : \mathfrak{I} \models d \in i\}$ gives a dense subset. By relativizing the quantifiers in A to the indices of non-empty levels, we in effect relativize to a finite subset of the domain. We make this more precise:

Suppose A is classically false in some finite structure $\mathfrak{I}$. W.l.o.g. we may assume that the domain of this structure is the naturals $0, \dots, n$. We extend $\mathfrak{I}$ to a $\mathbf{G}_V$-interpretation $\mathfrak{I}^g$ with domain $\mathbb{N}$ as follows: Since V contains infinitely many values, we can choose c_1, c_2, L and P so that $\exists x(x \in i)$ is true for $i = 0, \dots, n$ and false otherwise, and so that $\sup \mathrm{Distr}_{\mathfrak{I}^g} P(x) < 1$. The number-theoretic symbols receive their natural interpretation. The antecedent of A^g clearly receives the value 1, and the consequent receives $\sup \mathrm{Distr} \mathfrak{I}^g P(x) < 1$, so $\mathfrak{I}^g \not\models A^g$.

Now suppose that $\mathfrak{I} \not\models A^g$. Then $\mathfrak{I}(\exists x P(x)) < 1$ and so $\sup \mathrm{Distr}_{\mathfrak{I}} P(x) < 1$. In this case, $\mathfrak{I}(x \prec y) = 1$ iff $\mathfrak{I}(P(x)) < \mathfrak{I}(P(y))$, so $\prec$ defines a strict order on the domain of $\mathfrak{I}$. It is easily seen that in order for the value of the antecedent of A^g under $\mathfrak{I}$ to be greater than that of the consequent, it must be $= 1$ (the values of all subformulas are either $\leq \sup \mathrm{Distr}_{\mathfrak{I}} P(x)$ or $= 1$). For this to happen, of course, what the antecedent is intended to express must actually be true in $\mathfrak{I}$, i.e., that $x \in i$ defines a series of disjoint levels and that for any i, either level $i+1$ is empty or for all x, y s.t. $x \in j$, $y \in k$ with $j,k \leq i$ and $x \prec y$ there is a z with $x \prec z \prec y$ and $z \in i+1$. To see this, consider the relevant part of the antecedent, $B = \forall i \big[\forall x,y \forall j \forall k \exists z K \vee \forall x \neg(x \in i)\big]$. If $\mathfrak{I}(B) = 1$, then for all i, either $\mathfrak{I}(\forall x,y \forall j \forall k \exists z K) = 1$ or $\mathfrak{I}(\forall x \neg(x \in i)) = 1$. In the first case, we have $\mathfrak{I}(\exists z K) = 1$ for all x, y, j, and k. Now suppose that for all z, $\mathfrak{I}(K) < 1$, yet $\mathfrak{I}(\exists z K) = 1$. Then for at least some z the value of that formula would have to be $> \sup \mathrm{Distr}_{\mathfrak{I}} P(z)$, which is impossible. Thus, for every x, y, j, k, there is a z such that $\mathfrak{I}(K) = 1$. But this means that for all x, y s.t. $x \in j$, $y \in k$ with $j,k \leq i$ and $x \prec y$ there is a z with $x \prec z \prec y$ and $z \in i+1$.

In the second case, where $\mathfrak{I}(\forall x \neg(x \in i)) = 1$, we have that $\mathfrak{I}(\neg(x \in i)) = 1$ for all x, hence $\mathfrak{I}(x \in i) = 0$ and level i is empty.

Since V contains no dense subset, from some finite level i onward, the levels must be empty. Of course, $i > 0$ since $c_1 \in 0$. Thus, A is false in the classical interpretation $\mathfrak{I}^c$ obtained from $\mathfrak{I}$ by restricting $\mathfrak{I}$ to the domain $\{0, \dots, i-1\}$ and $\mathfrak{I}^c(Q) = \mathfrak{I}(\neg\neg Q)$ for atomic Q. □

This shows that no infinite-valued Gödel logic whose set of truth values does not contain a dense subset is axiomatizable. We strengthen this result to show that the prenex fragments are likewise not axiomatizable. This is done by showing that if A is prenex, then there is a formula A^G which is also prenex and which is valid in $\mathbf{G}_V$ iff A^g is. Since not all quantifier shifting rules are generally valid, we have to prove that in this particular instance there is a prenex formula which is valid in $\mathbf{G}_V$ iff A^g is.

Theorem 6.2 *If V is countably infinite, the prenex fragment of $\mathbf{G}_V$ is not axiomatizable.*

Proof. By the proof of Theorem 6.1, a formula A is true in all finite models iff $\mathbf{G}_V \models A^g$. A^g is of the form $B \to (A' \vee \exists u P(u))$. We show that A^g is equivalent in $\mathbf{G}_V$ to a prenex formula.

Call a formula A in which every atomic formula occurs negated a *classical* formula. It is easy to see that for any $\mathfrak{I}$ and $A(x)$ with $\mathfrak{I}(A(d)) \in \{0,1\}$ for all d, $\mathfrak{I}(\forall x A(x) \to B) = \mathfrak{I}(\exists x A(x) \to B)$ and $\mathfrak{I}(B \to \exists x A(x)) = \mathfrak{I}(\exists x(B \to A(x))$. Hence, any classical formula is equivalent to a prenex formula; let A_0 be a prenex form of A'. Since all quantifier shifts for conjunctions are valid, the antecedent B of A^g is equivalent to a prenex formula $\mathsf{Q}_1 x_1 \ldots \mathsf{Q}_n x_n B_0(x_1, \ldots, x_n)$. Hence, A^g is equivalent to $\overline{\mathsf{Q}}\bar{x} B_0(\bar{x}) \to (A_0 \vee \exists u P(u))$.

Let Q'_i be $\exists$ if Q_i is $\forall$, and $\forall$ if Q_i is $\exists$, let $C \equiv A_0 \vee \exists u P(u)$, and $v = \mathfrak{I}(\exists u P(u)))$. We show that $\overline{\mathsf{Q}}\bar{x} B_0(\bar{x}) \to C$ is equivalent to $\overline{\mathsf{Q}}'\bar{x}(B_0(\bar{x}) \to C)$ by induction on n. Let $\overline{\mathsf{Q}}\bar{x} B_0 \equiv \mathsf{Q}_1 x_1 \ldots \mathsf{Q}_i x_i B_1(d_1, \ldots, d_{i-1}, x_i)$. Since quantifier shifts for $\exists$ in the antecent of a conditional are valid, we only have to consider the case $\mathsf{Q}_i = \forall$. Suppose $\mathfrak{I}(\forall x_i B_1(\overline{d}, x_i) \to C) \neq \mathfrak{I}(\exists x_i(B_1(\overline{d}, x_i) \to C)$. This can only happen if $\mathfrak{I}(\forall x_i B_1(\overline{d}, x_i)) = \mathfrak{I}(C) < 1$ but $\mathfrak{I}(B_1(\overline{d}, c)) > \mathfrak{I}(C) \geq v$ for all c. However, it is easy to see by inspecting B that $\mathfrak{I}(B_1(\overline{d}, c))$ is either $= 1$ or $\leq v$.

Now we show that $\mathfrak{I}(B_0(\overline{d}) \to (A_0 \vee \exists u P(u))) = \mathfrak{I}(\exists u(B_0(\overline{d}) \to (A_0 \vee P(u))))$. If $\mathfrak{I}(A_0) = 1$, then both sides equal $= 1$. If $\mathfrak{I}(A_0) = 0$, then $\mathfrak{I}(A_0 \vee \exists u P(u)) = v$. The only case where the two sides might differ is if $\mathfrak{I}(B_0(\overline{d})) = v$ but $\mathfrak{I}(A_0 \vee P(c)) = \mathfrak{I}(P(c)) < v$ for all c. But inspection of B_0 shows that $\mathfrak{I}(B_0(\bar{t})) = 1$ or $= \mathfrak{I}(P(e))$ for some $e \in \overline{d}$ (the only subformulas of $B_0(\overline{d})$ which do not appear negated are of the form $e' \prec e$). Hence, if $\mathfrak{I}(B_0(\overline{d})) = v$, then for some e, $\mathfrak{I}(P(e)) = v$.

Last we consider the quantifiers in $A_0 \equiv \overline{\mathsf{Q}}\bar{y} A_1$. Since A_0 is classical, $\mathfrak{I}(B_0(\overline{d}) \to (A_0 \vee P(c))) = \mathfrak{I}(\overline{\mathsf{Q}}\bar{y}(B_0(\overline{d}) \to (A_1 \vee P(c))))$ for all $\overline{d}$, c. To see this, first note that shifting quantifiers across $\vee$, and shifting universal quantifiers out of the consequent of a conditional is always possible. Hence it suffices to consider the case of $\exists$. $\mathfrak{I}(\exists y A_2)$ is either $= 0$ or $= 1$. In the former case, both sides equal $\mathfrak{I}(B_0(\overline{d}) \to P(d))$, in the latter, both sides equal 1. □

In summary, we obtain the following characterization of axiomatizability of prenex fragments of Gödel logics:

Theorem 6.3 *The prenex fragment of $\mathbf{G}_V$ is axiomatizable if and only if V is finite or uncountable.*

7 Conclusion

Our characterization relates in an interesting way to compactness results of entailment relations of Gödel logics as given in [5]: Exactly those Gödel logics have an axiomatizable prenex fragment which also have a compact propositional logic.

For full first order Gödel logics the situation is quite similar in the sense that the truth value set must be finite or uncountable to allow axiomatization, but in addition it is necessary that 0 either be in the perfect kernel of the truth value set or be isolated. Two different logics correspond to these conditions, which have the same prenex fragment. Consequently there are Gödel logics where the prenex fragment is axiomatizable, but the full logic does not allow a recursive axiomatization. These are the logics of truth value sets which contain an uncountable subset, but 0 is neither in the perfect kernel nor isolated. These results have been obtained in [12] and will be reported in a forthcoming article by the authors.

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