An Introduction to Proof Theory

Revisions and Corrections
September 16, 2023

We are grateful to the many readers who reported typos, errors, and raised questions and provided suggestions for improvement, especially Yong Cheng, Alberto Naibo, Pierre Wagner, James Walsh, and David Waszek.

Preface

- p. viii, l. −13: add footnote to "simple induction on $\mathbb{N}$" reading: "Here, we mean inductive reasoning on the natural numbers; we will later (in section 2.7) distinguish between two forms thereof, successor induction and strong induction."
- p. ix, l. 9. "other elements" → "basic elements"
- p. ix, l. 19: "specific examples" → "specific cases"
- p. xi. Add to references in section 8:

  Arai, Toshiyasu. 2020. *Ordinal Analysis with an Introduction to Proof Theory*. Logic in Asia: Studia Logica Library. Singapore: Springer. DOI: 10.1007/978-981-15-6459-8.

Chapter 1

- p. 1, 1st para: Replace

  While reflection on mathematical proof goes as far back as the time of the ancient Greeks, it reached the late nineteenth century with the detailed formalization of mathematical proofs given first by Frege and Peano and later by Russell and Whitehead, and others. A mathematical analysis of proofs considered as mathematical objects only truly began with David Hilbert and his school.

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  Reflection on mathematical proof goes as far back as the time of the ancient Greeks and it reached the late nineteenth century with the detailed formalization of mathematical proofs given first by Frege and Peano and later by Russell and Whitehead, and others. However, the mathematical analysis of proofs considered as mathematical objects in their own right only truly began with David Hilbert and his school.

- p. 1, 2nd para, l. 8–9: "having its intended meaning" → "referring to"; l. 9–10: "a set of pairs" → "as a set of pairs"
- p. 3, 1st full para, l. 6: "the positive part" → "the second (and more positive) part"
- p. 3, fn. 3, l. 2: "of the domain" → "in the domain"
- p. 3, fn. 3, l. 5: "obtained as" → "obtained by taking"; fn. 3, l. −2: "that contain" → "that contains"
- p. 4, l. 6: "bais" → "basis"
- p. 4, fn 6, l. 2: "have a meaning associated to their use" → "rest on the meaning of the signs"
- p. 7, l. −4: "or follows from the one, two, or three formulas immediately above it in the tree by a rule of inference" → "is licensed by a rule of inference using the one, two, or three formulas above it"
- p. 10, l. 5: "finitist" → "finitary"
- p. 11, last para, l. 2: "proof theory is" → "proof theory is found in"

Chapter 2

- p. 13, 1st para, l. −5: "system" → "systems"; 1st para, l. −2: "showing" → "proving"
- p. 14, 1st para, l. 2: "a class" → "a class (or set)"; l. 5: "clause" → "clause or clauses"; l. −1: "clause(s)" → "clause (or clauses)"
- p. 14, 2nd para, l. −3: "The extremal clause would ensure that we only take the smallest class containing Frege's father and mother and that is closed under the functions mother(x) and father(x)" → "The extremal clause guarantees that we only take the smallest class containing Frege's mother and father and that is closed under the functions mother(x) and father(x)"
- p. 14, 3rd para, l. 2: "set" → "class"
- p. 15, 2nd para, l. 4: "collection" → "class"
- p. 15, 2nd para, l. −7: "inductive clauses" → "inductive clause"

- p. 17, 3rd para, l. 1: "every formula" → "every non-atomic formula"
- p. 18: Definition 2.5: Delete clause (3).
- p. 18, l. −1 to p. 19, l. 2: "Each calculus is stronger than the previous one(s), and it is obtained by adding more powerful assumptions to its predecessor." → "Each calculus is given by a list of axioms and inference rules. The inference rules are common to all three systems. The axioms of our systems are obtained from previous ones by adding additional axioms, making $\mathbf{J}_0$ stronger than $\mathbf{M}_0$, and $\mathbf{K}_0$ stronger than $\mathbf{J}_0$."
- p. 19, Section 2.4.1, 1st para: Delete
- p. 19, Section 2.4.1, 2nd para, l. 1: "We will first present the systems in a purely formal way" → "We will first present the systems in a purely formal way by listing their axioms."
- p. 19, after list of axioms: "givethe" → "reproduce the"
- p. 19, just before section 2.4.2: insert:

  > **Remark.** The axioms, as you may have noticed, are not strictly speaking formulas of the propositional language, but contain metavariables (cf. Remark on p. 14). They are axiom schemas, each describing its infinitely many instances. For example, any formula that has the form $A \supset (A \land A)$ counts as an axiom of type PL1. It is understood that every formula of this form counts as an axiom, such as $p_1 \supset (p_1 \land p_1)$, $\neg(p_1 \lor p_2) \supset (\neg(p_1 \lor p_2) \land \neg(p_1 \lor p_2))$, etc. We do this to avoid the need to include a substitution rule (which is what Gentzen and Heyting did). In what follows, for simplicity, we will not always carefully distinguish between the schemas and their instances and let context disambiguate whether we mean the schema or its instances.

- p. 20, 4th para, l. 1: Delete: "But in general we will prove schematic theorems that go proxy for an infinite number of proofs of their instances."
- p. 20: remove turnstiles in the displayed derivation
- p. 20, after the derivation, add: "Thus, the formula in line 11 is a theorem. We record this fact by writing $\vdash p_1 \supset (p_2 \lor p_1)$."
- p. 20, fn. 9: Delete
- p. 21, 1st para, l. −4: "furmula" → "formula"
- p. 21, 1st para, l. −2: Insert comma: "$p_1 \supset (p_2 \lor p_1)$, is the end-formula"

- p. 21, 2nd para, l. 2: "axiom" → "axiom schema"; l. 3: "axiom" → "schema"; l. 4: "We note which axiom a formula is an instance of on the right" → "To the right of each line, we note the axiom schema of which the formula on that line is an instance"

- p. 21 "The first thing. . . " to p. 22 just before "Note that since" Replace with:

  The formulas in our example derivation involve the propositional variables p_1 and p_2. If we uniformly replace p_1 and p_2 in our derivation by arbitrary formulas C and D, respectively, we still have a correct derivation. This is because such a uniform substitution turns axioms into axioms, and valid applications of MP into valid applications of MP. On line 1, $p_1 \supset (p_1 \lor p_2)$ is an instance of schema PL7, but any formula of the form $C \supset (C \lor D)$ is also an instance of schema PL7. The result of replacing p_1 and p_2 by C and D on line 2, the formula

  $$[C \supset (C \lor D)] \supset [((C \lor D) \supset (D \lor C)) \supset (C \supset (C \lor D))],$$

  is also an instance of $B \supset (A \supset B)$, so line 2 will still be an axiom because it is an instance of schema PL5. If we now abbreviate $C \supset (C \lor D)$ by A, and $((C \lor D) \supset (D \lor C)) \supset (C \supset (C \lor D))$ by B, we see that lines 1 and 2 are of the forms A and $A \supset B$, respectively, so line 3 follows from 1 and 2 by MP. For this reason, we can generally write our derivations with schematic letters rather than propositional variables. Our first derivation would then turn into a schematic proof of $C \supset (D \lor C)$ which shows that not only is $p_1 \supset (p_2 \lor p_1)$ a theorem, but any formula of the form $C \supset (D \lor C)$ is.

  In practice we will not actually write out complete derivations, but rather record how derivations will be constructed from derivations we've already given, making use of MP and what we'll call *derived rules*. Consider the first three lines of our derivation, where we abbreviate $C \supset (C \lor D)$ by E and $(C \lor D) \supset (D \lor C)$ by F. We then have:

1.	E	(axiom PL7)
2.	$E \supset (F \supset E)$	(axiom PL5)
3.	$F \supset E$	(MP 1, 2)

  In our example, line 1 is an instance of an axiom schema, but it need not be: If E is any theorem, we can use a derivation with E as its end-formula, then add lines 2 and 3, and obtain a derivation of $F \supset E$, for any formula F. We can record this as follows:

1.	$\vdash E$	(hypothesis)
2.	$\vdash E \supset (F \supset E)$	(axiom PL5)
3.	$\vdash F \supset E$	(MP 1, 2)

Line 1 now is not an axiom, but the result of a derivation, i.e., a theorem. We indicate this by prefixing each line with the derivability symbol $\vdash$. We have shown that if E is a theorem, then $F \supset E$ also is a theorem. We call this a *schematic (meta)derivation*.

We can use the first 9 lines of our derivation to prove something more:

1.	$\vdash E$	(hypothesis)
2.	$\vdash E \supset (F \supset E)$	(axiom PL5)
3.	$\vdash F \supset E$	($\textsc{mp}$ 1, 2)
4.	$\vdash (F \supset E) \supset [(F \wedge F) \supset (E \wedge F)]$	(axiom PL3)
5.	$\vdash (F \wedge F) \supset (E \wedge F)$	($\textsc{mp}$ 3, 4)
6.	$\vdash F \supset (F \wedge F)$	(axiom PL1)
7.	$\vdash F$	(hypothesis)
8.	$\vdash F \wedge F$	($\textsc{mp}$ 6, 7)
9.	$\vdash E \wedge F$	($\textsc{mp}$ 5, 8)

This schematic derivation shows that if E and F are both theorems, so is $E \wedge F$, or for short: If $\vdash_{\mathbf{M}_0} E$ and $\vdash_{\mathbf{M}_0} F$, then $\vdash_{\mathbf{M}_0} E \wedge F$. This is our first *derived rule:*

$\wedge$INTRO: If $\vdash_{\mathbf{M}_0} A$ and $\vdash_{\mathbf{M}_0} B$, then $\vdash_{\mathbf{M}_0} A \wedge B$.

We will be able to appeal to this rule to show in $\mathbf{M}_0$ (and in the other calculi as well), that if we have derived the formulas A and B, we can derive $A \wedge B$ as well.

In our derivations we will often need to make use of the following strategy. Suppose we have that $\vdash_{\mathbf{M}_0} A \supset B$ and $\vdash_{\mathbf{M}_0} B \supset C$. Intuitively, it should be possible to prove that $\vdash_{\mathbf{M}_0} A \supset C$ as well. But modus ponens is our only rule, and does not allow us to make this inference directly. But we can now make use of our derived rule $\wedge$INTRO, and provide the following schematic derivation:

1.	$\vdash A \supset B$	(hypothesis)
2.	$\vdash B \supset C$	(hypothesis)
3.	$\vdash (A \supset B) \wedge (B \supset C)$	($\wedge$INTRO 1, 2)
4.	$\vdash [(A \supset B) \wedge (B \supset C)] \supset (A \supset C)$	(axiom PL4)
5.	$\vdash A \supset C$	($\textsc{mp}$ 3, 4)

We can think of this schematic derivation as establishing the property of transitivity for $\supset$. It is our second derived rule:

$\supset$TRANS: If $\vdash_{\mathbf{M}_0} A \supset B$ and $\vdash_{\mathbf{M}_0} B \supset C$, then $\vdash_{\mathbf{M}_0} A \supset C$.

- p. 23, 1st para, l. 6: “schematic routine derivations” → “routine schematic derivations”; l. −6: “result” → “results”

- p. 25, 3rd para, l. 3: "*provable* (in the appropriate system)" → "*provable* (in the appropriate system) in the sense of Definition 2.6"
- p. 25, item (2) in list: "can also be proved" → "can also be derived"
- p. 26, Section 2.7, 2nd para, l. −4: "is either even or odd" → "is even or odd"
- p. 28, l. −9: "Suppose the proof of the inductive step involves just one $m_1 < n$." → "Suppose that in the proof of the inductive step, the proof of $P(n)$ only rests on the hypothesis $P(m_1)$ for a single $m_1 < n$."
- p. 29, 2nd para, l. 2: "other kinds of objects" → "other kinds of objects that involve also non-arithmetical notions (for instance, syntactic notions such as formulas, derivations, etc.)"
- p. 29, 2nd para, l. 6; 3rd para, l. 3; 4th para, l. 1; 5th para, l. 1, l. 2; 6th para, l. 1, l. 2 : "in stage" → "at stage"; 2nd para, l. 7: "in the" → "at the"; 6th para, l. 4: "in some stage stage" → "at some stage"
- p. 30, Problem 2.15, (2) of second definition: "except when k_1 and k_2 are both 0" → "except when k_1 and k_2 are both 0, in which case it is 0"
- p. 32, delete period at end of third displayed formula (l. 12)
- p. 32, replace

  > One example will be sufficient to show how much easier it is now to prove conditional theorems. Suppose we want to prove
  >
  > $$[A \wedge ((A \wedge B) \supset C)] \supset (B \supset C)$$
  >
  > Assume we have reached
  >
  > $$\{A \wedge ((A \wedge B) \supset C)\} \cup \{B\} \vdash C$$
  >
  > by means of the following derivation:

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  > Suppose we want to prove
  >
  > $$[A \wedge ((A \wedge B) \supset C)] \supset (B \supset C)$$
  >
  > First we establish that
  >
  > $$\{A \wedge ((A \wedge B) \supset C)\} \cup \{B\} \vdash C$$
  >
  > by means of the following strategy:

- p. 33, replace first paragraph with:

  Here we extend our previous convention for recording facts about derivability ("schematic derivations") to derivations from assumptions. Lines 1 and 2 are not theorems, but assumptions to be used later. For this reason, they are not prefixed by $\vdash$. Line 3 *is* a theorem. Line 4 records that from the assumption on line 1 we can derive A, i.e., there is a derivation that shows $\{A \wedge ((A \wedge B) \supset C\} \vdash A$.

- p. 34, 1st full para, l. 2–3: "initial nodes" → "topmost nodes (leaves)"
- p. 34, 1st full para, l. 3: "derivations" → "derivation"
- p. 34, 2nd para, l. 2: "the" → "a"
- p. 34, 3rd full para, l. 1: "need to single out by a different node the repeated use of $A \supset B$" → "repeated use of $A \supset B$ in two different nodes"
- p. 35, 2nd para, l. 2: "when" → "where"
- p. 35, 2nd para, l. 3; l. 5: "premises" → "assumptions"
- p. 36, 3rd full para, l. 3: "the assumption A leads to a contradiction" → "turns a construction of A into a construction of the contradiction"
- p. 37, l. 3: "is equivalent" → "as equivalent"
- p. 37, l. 7: "an effective proof" → "a proof"
- p. 37, 3rd para, l. 4: "the proof" → "a proof"; l. 6: "fact" → "principle"
- p. 38, Section 2.10.2, 1st para, l. 3: ".." → "."
- p. 40: Replace 2nd display by:

  $$\vdash_{\mathbf{J}_0} (A \vee B) \supset \neg(\neg A \wedge \neg B)$$
  $$\vdash_{\mathbf{J}_0} (A \wedge B) \supset \neg(\neg A \vee \neg B)$$

- p. 40, Section 2.11, replace beginning of section (to just before "As a consequence" at the end of p. 41) with:

  We have confidently asserted that $\neg A \supset (A \supset B)$ cannot be proved in $\mathbf{M}_0$ and that $\neg\neg A \supset A$ cannot be proved in $\mathbf{J}_0$. But how do we know that? In this section we want to explain one of the techniques for proving such results. The technique is model-theoretic (as opposed to proof-theoretic) and consists in showing that we can concoct interpretations (or models) of our axiom systems that verify the axioms and preserve the validity of the inferential rules but falsify the statement we want to show to be independent. The basic idea is that we can consider connectives as function symbols operating on "truth-values." These function symbols output certain values depending on the input value of their arguments according to a specific table of values.

The systems look very similar to the truth-tables familiar from a first logic course, except that in many cases, instead of having only two truth-values (0, 1 or T, F) we might have more than two, and occasionally more than one truth-value can play the role that T (or F) played for truth-tables.

Let us first prove that $\neg A \supset (A \supset B)$ and $\neg\neg A \supset A$ are not derivable in $\mathbf{M}_0$. Consider the following truth-tables for $\supset, \wedge, \vee, \neg$ where we have only two truth-values (0, 1) and 0 plays the role of truth.[1] (When using the table for a binary connective, the first argument is given by the leftmost vertical column and the second argument by the horizontal row at the top.)

$\supset$	0	1
0	0	1
1	0	0

$\wedge$	0	1
0	0	1
1	1	1

$\vee$	0	1
0	0	0
1	0	1

$\neg$	
0	0
1	0

Notice that the interpretation is purely algebraic.[2] For instance, it certainly does not capture our intuition about negation to see that whether the input is 0 or 1, our table always outputs 0. But that does not matter, as we are only interested in showing the possibility of an interpretation of the connectives as functions over truth-values which will show that the value of the two propositions is "falsity" while the other axioms and their consequences can only be "true."

If we assign 0 or 1 to the propositional variables in a formula, the truth-tables above determine a value (again, either 0 or 1) for the entire formula. A formula that always takes the value 0 then counts as a tautology. If a formula is a tautology, so is every substitution instance of it. For instance, if $p_1 \supset (p_1 \wedge p_1)$ is a tautology, so is every formula of the form $A \supset (A \wedge A)$, i.e., every instance of schema PL1.

We now show, by induction on the length of proofs, that every formula provable in $\mathbf{M}_0$ is a tautology in this non-standard sense. If we can show that a formula doesn't have this property (it takes value 1 for some assignment), it cannot be derivable in $\mathbf{M}_0$. You should verify that all axioms of $\mathbf{M}_0$ always have value 0 according to the above tables, and further, that modus ponens always lead from premises with values 0 to a conclusion with value 0, i.e., if A and $A \supset B$ have value 0, B must also have value 0.

Now consider the instances $\neg p_1 \supset (p_1 \supset p_2)$ and $\neg\neg p_1 \supset p_1$ of our two formulas. They are not tautologies; by a suitable assignment of 0 and 1

[1] The assignment of the numbers 0 as "true" and 1 (and later also 2) as "false" is purely conventional; often 1 is used as "true" and 0 as "false."

[2] In this case, all the connectives except negation are interpreted as in classical logic.

to p_1 and p_2 we can obtain the value 1. For the first formula, assign 0 to p_1 and 1 to p_2. With these values assigned to p_1 and p_2, we have that $p_1 \supset p_2$ has value 1, $\neg p_1$ has value 0, and $\neg p_1 \supset (p_1 \supset p_2)$ has value 1. To show the possible "falsity" of the second formula, instead assign 1 to p_1. Then $\neg\neg p_1$ takes the value 0 and so $\neg\neg p_1 \supset p_1$ has value 1. So, neither $\neg p_1 \supset (p_1 \supset p_2)$ nor $\neg\neg p_1 \supset p_1$ can have a derivation in $\mathbf{M}_0$. Consequently, there can be no schematic derivation of $\neg A \supset (A \supset B)$ or of $\neg\neg A \supset A$.[3]

- p. 42. Replace Problem 2.21 and the paragraph following by:

  Problem 2.21 Let $\mathbf{M}_0^+$ be the calculus obtained from $\mathbf{M}_0$ by adding the axiom schema $A \lor \neg A$. Using the first set of tables above, show that $\nvdash_{\mathbf{M}_0^+} \neg\neg A \supset A$.

  In the previous section we showed that $\vdash_{\mathbf{K}_0} A \lor \neg A$. This proof required axiom schema PL12 ($\neg\neg A \supset A$), but did not use schema PL11, so in fact $A \lor \neg A$ is already provable in the calculus resulting from $\mathbf{M}_0$ by adding $\neg\neg A \supset A$ as an axiom schema. The preceding problem thus shows that in a sense schema $\neg\neg A \supset A$ is stronger than $A \lor \neg A$.

- p. 45, l. 2: "variables, and" → "variables, f, g, h as metavariables for function symbols, and"

- p. 45, l. 3: add at end of para: "In addition to A, B, C, etc., as metavariables for formulas (see definition below), in the following we will also use F, G, etc., and $F(a)$, $G(a)$, etc., to vary over sentences and formulas with one free variable. Occasionally we will also use P, Q with or without variables to also stand for formulas."

- p. 45, Definition 2.30, (2): "Inductive clause" → "Inductive clauses"

- p. 48, 3rd para, l. 4: "in the axioms by formulas" → "in the axiom schemas by schematic formulas"; l. 5: "For instance, we have $\vdash_{\mathbf{M}_1} F(a) \supset (F(a) \land F(a))$ by PL1." → "For instance, we have $\vdash_{\mathbf{M}_1} F(a) \supset (F(a) \land F(a))$, for any formula $F(a)$ with a free variable a, by PL1."

- p. 49, just before Definition 2.36: "premises Γ" → "assumptions Γ"

- p. 50, first para, l. 2–5: "Assume $B \in \Gamma$ and let $A_1 \ldots A_n$ be a derivation from Γ with the justifications for each step. A_i *depends on* B if, and only if, B is the formula A_i and the justification for B is that it belongs to Γ; or, there are A_k, A_ℓ with $k, \ell < i$ such that A_i is justified by an application of modus ponens, QR1 or QR2 applied to A_k, A_ℓ and at least one of A_k, A_ℓ depends on B." →

[3] Note that for *some* instances there may well be derivations; we just can't have derivations for *all* instances. E.g., $\neg\neg(p_1 \supset p_1) \supset (p_1 \supset p_1)$ is derivable, since $p_1 \supset p_1$ is a theorem, and whenever A is a theorem so is $B \supset A$ for any formula B.

> Assume $B \in \Gamma$ and let $A_1 \ldots A_n$ be a derivation from Γ with the justifications for each step given. A_i *depends on* B if, and only if, one of the following holds:
>
> 1. B and A_i are the same formula, and the justification for A_i is that it (i.e., B) belongs to Γ;
> 2. there are A_k, A_ℓ with $k, \ell < i$ such that A_i is justified by an application of modus ponens and at least one of A_k, A_ℓ depends on B;
> 3. there is an A_k with $k < i$ such that A_i is justified by an application of QR$_1$ or QR$_2$ applied to A_k and A_k depends on B.

- p. 50, proof of Lemma 2.37, l. 3: "A can only be an axiom or an element of Γ which is not B" $\to$ "A can only be an axiom or an element of Γ which is not B (otherwise it would depend on B by (1) above, contrary to the assumption that A does not depend on B)"

- p. 50, proof of Lemma 2.37, l. -2: "corresponding rules" $\to$ "corresponding rule"

- p. 51, proof of Theorem 2.38, inductive step, l. 1: Replace

  > We have four cases. A_n could be B; A_n could be in Γ and different from B; A_n could be an axiom; A_n could be obtained from previous formulas A_k, A_ℓ (where $k, \ell < n$) by MP, QR$_1$, or QR$_2$. The first three cases are dealt with as in the case for $n = 1$. The last case contains three sub-cases depending on the rule of inference used to derive A_n. We will treat MP and QR$_1$, leaving QR$_2$ as an exercise."

  by

  > We have six cases. A_n could be B; A_n could be in Γ and different from B; A_n could be an axiom; A_n could be obtained from previous formulas A_k, A_ℓ (where $k, \ell < n$) by MP; A_n could be obtained from a previous formula A_k (where $k < n$) by QR$_1$; or it could be obtained from some A_k by QR$_2$. The first three cases are dealt with as in the case for $n = 1$. Of the last three cases will treat MP and QR$_1$, leaving QR$_2$ as an exercise.

- p. 51, proof of Theorem 2.38, second para: "Let us assume $\Gamma, B \vdash A_n$ is obtained from $\Gamma, B \vdash A_k$ and $\Gamma, B \vdash A_\ell$ where A_k has the form $A_\ell \supset A_n$ and where each derivation satisfies the hypothesis of the theorem." $\to$

  > Let us assume that A_n is justified by MP applied to A_k and A_ℓ, where A_k has the form $A_\ell \supset A_n$. The derivations ending in

> A_k and A_ℓ justify, respectively, that $\Gamma, B \vdash A_k$ and $\Gamma, B \vdash A_\ell$. Clearly, each derivation satisfies the hypothesis of the theorem, as any QR$_1$ and QR$_2$ inferences in them are also inferences in the derivation of $A = A_n$ from Γ, B itself.

- p. 52, l. 1: "Assume now that $\Gamma, B \vdash A_n$ where A_n is obtained by QR$_1$ from $\Gamma, B \vdash A_\ell$, the conclusion A_ℓ has the form $D \supset C(a)$, and A_n has the form $D \supset \forall x\, C(x)$." $\longrightarrow$

  > Assume now that A_n is justified by QR$_1$ applied to A_ℓ. The derivation ending in A_ℓ shows that $\Gamma, B \vdash A_\ell$, and again this derivation satisfies the assumption of the theorem. The conclusion A_ℓ has the form $D \supset C(a)$, and A_n has the form $D \supset \forall x\, C(x)$.

- p. 52, l. 3: "$\Gamma, B \vdash A_n$" $\longrightarrow$ "A_n"
- p. 53, 1st para, l. 5: "sequent calculi" $\longrightarrow$ "the sequent calculus"
- p. 53, 4th para, l. 1: "sets in which the" $\longrightarrow$ "sets for which"
- p. 54, 2nd full para, l. 2: "this chapter" $\longrightarrow$ "this section"; 4th para, l. 1: "no one doubted" $\longrightarrow$ "virtually no one thought"; "finitistically and" $\longrightarrow$ "finitarily or"
- p. 55, l. 5: "drawn" $\longrightarrow$ "generally drawn"
- p. 55, section 2.15.2, 1st para, l. -3: "constant" $\longrightarrow$ "individual constant"
- p. 56, Definition 2.42, (2): "Inductive clause" $\longrightarrow$ "Inductive clauses"
- p. 57, first displayed formula: "$\exists x\, F(x) \supset \exists x \forall y(y < x \supset \neg F(y))$" $\longrightarrow$ "$\exists x\, F(x) \supset \exists x(F(x) \land \forall y(y < x \supset \neg F(y)))$"
- p. 58, l. -1: "$\mathbf{J}_1$" $\longrightarrow$ "$\mathbf{PA}_I$"
- p. 59, Definition 2.46: Add footnote: "The translation of Definition 2.46 and the analog of Theorem 2.48 also works for formulas of predicate logic, not just formulas and proofs in arithmetic. In that case it is necessary to define $G^* = \neg\neg G$ for G atomic."
- p. 62, derivation line 6: "$\neg\forall x\, B^*(x) \supset \neg\neg A^*$" $\longrightarrow$ "$\neg\forall x\, \neg B^*(x) \supset \neg\neg A^*$"
- p. 62, l. -3: "derivation of" $\longrightarrow$ "derivation"; l. -1: "is" $\longrightarrow$ "is a"
- p. 63, l. -3 "$\neg(0 = 0)$" $\longrightarrow$ "$\neg 0 = 0$"
- p. 64, l. 1: "it is inconsistent" $\longrightarrow$ "T is inconsistent"

Chapter 3

- p. 69, l. −2: “sub-proof(s)” → “sub-deduction(s)”
- p. 70, first full para, l. −1: “given in Table 3.1” → “is given in Table 3.1 on p. 88.”
- p. 75, first para, l. −3: “new” → “new application of the”; l. −2: “(lowermost)” → “(lowermost) application of the”
- p. 75, 2nd para, l. −5: “deductions” → “deduction”
- p. 76, 1st para, l. −1: “labelled ^{1}” → “labelled 1”
- p. 77, 1st para, l. 1, 2, 4: “sub-proof” → “sub-deduction”
- p. 77, Section 3.2.4, 3rd para, l. 2: “$B \equiv \perp$” → “$B = \perp$”
- p. 78, 1st full para, l. −3: “A simple example” → “For a simple example”
- p. 79, section 3.2.5, 2nd para, l. −1: “A free variable so restricted is called the *eigenvariable* of the inference.” → “The indicated free variable c which is so restricted is called the *eigenvariable* of the inference.”
- p. 82, replace

  > Of course, not any free variable would be suitable to use for this purpose: only variables about which we haven’t already assumed something, i.e., only variables that don’t appear in the major premise or any open assumptions on which it or the subsidiary conclusion C depends. These rather complicated preconditions on the use of the variables c are expressed in a restriction on the use of $\exists$E: Only applications of $\exists$E are allowed where the variable c does not appear in C or in any assumptions that are open after the application of the inference. This means in particular that c may not be open in an assumption of the sub-deduction leading to the major premise $\exists x\, A(x)$, in any assumption not of the form $A(c)$ in the sub-deduction leading to the minor premise C, and that *all* open assumptions of the form $A(c)$ in that right sub-deduction must be discharged at this inference.

  by

  > Of course, not any free variable would be suitable to use for this purpose: only variables about which we haven’t already assumed something other than that $A(c)$ holds. This rather vague condition on the use of the variable c is expressed in a

restriction on the use of $\exists$E: Only applications of $\exists$E are allowed where the variable c does not occur in $\exists x\, A(x)$, C, or in any assumptions that are open in the sub-deduction ending in the minor premise C other than the assumptions of the form $A(c)$ being discharged by the inference.

- p. 82, 1st para, l. -3: Delete "At an application of $\exists$E, any assumptions of the form $A(c)$ which are open in the deduction ending in the minor premise may be discharged."
- p. 82, add at the end of the page:

  To see what can go wrong if c were allowed to occur in an open assumption other than $A(c)$ on the right, consider the deduction

$$
\dfrac{\exists x\, A(x) \qquad \dfrac{A(c)^{1} \qquad A(c) \supset C}{C}\supset\mathrm{E}}{C}\,\exists\mathrm{E}\;{}_{1}
$$

  This is a deduction of C from the open assumptions $\exists x\, A(x)$ and $A(c) \supset C$, but C does not follow from $\exists x\, A(x)$ and $A(c) \supset C$. Here's a "deduction" of an invalid formula where the condition that c must not occur in the major premise $\exists x\, A(x)$ is violated:

$$
{}_{1}\,\dfrac{{}_{2}\,\dfrac{\exists x\, A(x,c)^{1} \qquad \dfrac{A(c,c)^{2}}{\exists x\, A(x,x)}\exists\mathrm{I}}{\exists x\, A(x,x)}\exists\mathrm{E}}{\exists x\, A(x,c) \supset \exists x\, A(x,x)}\supset\mathrm{I}
$$

  Note that we do not require, as we also don't in $\supset$I and $\lor$E, that an application of $\exists$E must in fact discharge open assumptions of the form $A(c)$ on the right. If it does not, however, the application of $\exists$E is *redundant:* we could then deduce the conclusion C without its use (and in fact without the major premise $\exists x\, A(x)$), and with (possibly fewer) open assumptions, using just the sub-deduction ending in the premise C.

- p. 88, section 3.4. Label of inference rule on the right in the last display should read "ND"
- p. 89, replace the last deduction and the three lines below it by:

$$
\dfrac{\dfrac{{}_{1}\,\dfrac{A^{1}}{\neg A, A}\neg\mathrm{I}}{A \lor \neg A, A}\lor\mathrm{I}}{A \lor \neg A}\lor\mathrm{I}
$$

> The last $\vee$I inference results in the set $\{A \vee \neg A, A \vee \neg A\}$, which is identical to $\{A \vee \neg A\}$—if a rule yields as conclusion a formula already present, the two copies are automatically "contracted" into one.

- p. 89, section 3.5, 1st para, l. −4: "proof-theoretic" → "Proof-theoretic"
- p. 90, Definition 3.13, inductive clause: "ending in these premises" → "ending in the premises of the last inference"
- p. 90, last display: The conclusion "B" and the formula "A" on the right should be aligned at the baseline
- p. 91, l. 3: "premise" → "premise of the original deduction"
- p. 93, l. 2: "$(\forall x\, A(x))[c/a]$" → "$\forall x\, A(x)[c/a]$"
- p. 95, the $\exists$E inference in the second display is missing the discharge label 1:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\ \delta_1\\ \exists x\, A(x)\end{array} \qquad \begin{array}{c}\Gamma_2, A(c)^{1}\\ \vdots\ \delta_2\\ C\end{array}}{C}\ {\scriptstyle 1}\ \exists\mathrm{E}$$

- p. 95, in the proof of Lemma 3.18, replace the beginning of the last paragraph

> Let d again be a free variable which is not used as an eigenvariable in δ'_1 or δ'_2, which is not in V, and which does not occur in Γ_1, Γ_2 or C. As before, $\delta'_2[d/c]$ is a correct deduction of C from $\Gamma_2 \cup \{A(d)\}$ (by the eigenvariable condition, c does not occur in C or Γ_2). d does not occur in Γ_1, Γ_2, or C. So the eigenvariable condition for an application of $\exists$E is satisfied, and we can take as δ' the deduction

  with:

> Let d again be a free variable which is not used as an eigenvariable in δ'_1 or δ'_2, which is not in V, and which does not occur in $\exists x\, A(x)$, Γ_1, Γ_2 or C. As before, $\delta'_2[d/c]$ is a correct deduction of C from $\Gamma_2 \cup \{A(d)\}$: by the eigenvariable condition, c does not occur in C, Γ_2, or $\exists x\, A(x)$. Replacing c by d in δ_2 leaves assumptions in Γ_2 as well as C unchanged, and changes all assumptions of the form $A(c)$ into $A(d)$. Consequently, d does not occur in $\exists x\, A(x)$, C, or Γ_2, and the eigenvariable condition

for an application of $\exists$E is satisfied. We can take as δ' the deduction

- p. 95, the $\exists$E inference in the third display is missing the discharge label 1:

$$
{}_1\,\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\,\delta_1'\\ \exists x\, A(x)\end{array}\qquad \begin{array}{c}\Gamma_2, A(d)^{\,1}\\ \vdots\,\delta_2'[d/c]\\ C\end{array}}{C}\,\exists\mathrm{E}
$$

- p. 97, 2nd para, l. -2: "deduction" $\rightarrow$ "proof"; l. -1 "deduction" $\rightarrow$ "proof"
- p. 99, below the first display: "size" $\rightarrow$ "sizes"
- p. 99, below the third display: full stop for comma: "$(B \wedge (B \supset A)) \supset A$," $\rightarrow$ "$(B \wedge (B \supset A)) \supset A$."

Chapter 4

- p. 103, l. 2–3: Delete: "However, in general, indirect deductions can be *much* shorter than the shortest direct deduction." (It is repeated below)
- p. 104, l. 1: "to ensure δ_1 and δ_2 have no discharge labels in common)" $\rightarrow$ "to ensure that all discharge labels in the resulting deduction are distinct)"
- p. 105, 1st full para, l. 4: "normal deduction" $\rightarrow$ "normal deduction (from Γ)"
- p. 107, proof of Proposition 4.2, 1st para, l. -2: "for formulas of any given complexity" $\rightarrow$ "which have formulas of a given complexity as the conclusion"
- p. 108, 109: replace "$\equiv$" by "$=$" (8 times)
- p. 109, l. 3: "$B \equiv A \vee B$" $\rightarrow$ "$B \equiv C \vee D$"
- p. 109, 2nd full para, l. 2: "following $\bot_J$" $\rightarrow$ "including and following the last $\bot_J$ inference"
- p. 110, Definition 4.5, after the list: "our last example" $\rightarrow$ "our last example (on p. 102)"
- p. 111, 2nd para, l. 5: "atomic CUTS" $\rightarrow$"atomic cuts"
- p. 112, Problem 4.7, l. 2 and after the displayed deduction: "avoid" $\rightarrow$ "eliminate"

- p. 113, proof of Lemma 4.8: delete first paragraph ("We will assume ... labelled i in δ_2.")
- p. 113, proof of Lemma 4.8: Replace 4th para:

  > Now suppose B is an open assumption of the form A labelled i. Since B is the only open assumption of δ_1, this means B and A are identical formulas. Then $\delta_1[\delta_2/A^i]$ is δ_2, which is then also a deduction of B. Its open assumptions are those of δ_2, i.e., contained in $\Gamma_1 \cup \Gamma_2$. Trivially, A is open in $\delta_1[\delta_2/A^i] = \delta_2$ only if it is open in δ_2.

  by

  > Now suppose B is an open assumption of the form A labelled i. Since B is the only open assumption of δ_1, B and A must be identical formulas. Hence δ_2 is also a deduction of B. Let δ_2' be a deduction resulting from δ_2 by renumbering all discharge labels so that overall in the resulting deduction no two inferences use the same discharge label. In particular, this ensures that no assumption of the form A in δ_2' is labelled i. We let $\delta_1[\delta_2/A^i]$ be δ_2'. Its open assumptions are those of Γ_2, and thus are contained in $\Gamma_1 \cup \Gamma_2$. Trivially, A is open in $\delta_1[\delta_2/A^i] = \delta_2'$ only if it is open in δ_2.

- p. 113–2, proof of Lemma 4.8, inductive step: Replace "*I*" by "*Inf*" throughout (10 times)
- p. 115, l. 2 below the second display: after "rightmost, topmost cuts first" insert footnote: "The rightmost topmost cut is the cut that appears furthest to the right of all the cuts such that the deduction contains no cuts above them. Since of any two cuts, either one is above the other, or one is further to the right than the other, it is clear that the rightmost, topmost cut is unique."
- p. 116, Definition 4.9, l. 2: "*rank*" → "*cut rank*"
- p. 117, proof of Theorem 4.10, 3rd para, l. 4: "premise" → "major premise"
- p. 119, l. 6 below the first display: "the same as that of δ" → "the same as that of δ_1"; l. 9: "are introducted" → "are introduced in δ_2"
- p. 120, l. 2 below the first displayed deduction: "cut of degree 2" → "cut of degree 2 (indicated by a box above)"
- p. 120, l. −4 of the 1st para: "topmost, rightmost" → "rightmost, topmost"

- p. 121, l. 4: “topmost cut” → “topmost cut of maximal degree”

- p. 122, Definition 4.13, replace by:

  1. *Basis clause:* If A is atomic, A is the only sub-formula of A.
  2. *Inductive clauses:*
     a) If A is of the form $\neg B$, the sub-formulas of A are A itself, $\perp$, and the sub-formulas of B.
     b) If A is of the form $(B \land C)$, $(B \lor C)$, or $(B \supset C)$, then the sub-formulas of A are A itself and the sub-formulas of B and C.
     c) If A is of the form $\exists x\, B(x)$ or $\forall x\, B(x)$, and $B(x)$ is $B[x/a]$, then the sub-formulas of A are A itself and the sub-formulas of all formulas $B[t/a]$, t any term.
  3. *Extremal clause:* Nothing else is a sub-formula of A.

- p. 123, Proposition 4.16, l. 2 and l. 1 of proof: “$j > i$” → “$j \geq i$”

- p. 124, l. −3 and −2: delete “major” (twice)

- p. 124, l. −1: “by $\land$E” → “by $\land$E or $\forall$E”

- p. 129, l. −8, “formulas in the threads” → “formulas on the threads”

- p. 130, l. 1: “In our final example above” → “In our previous example on p. 128”

- p. 130, section 4.5, 1st para, l. 2–3: “they can however be more complex in that they are *much longer*” → “they are in general *much longer*”; second para, l. 3: “2^n” → “2^{n-1}”

- p. 131, replace the 4th displayed deduction by

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \delta\\ (B \supset C) \supset ((C \supset D) \supset (B \supset D))\end{array} \qquad \begin{array}{c}\vdots\ \delta_1\\ B \supset C\end{array}}{(C \supset D) \supset (B \supset D)}\,{\supset}\mathrm{E} \qquad \begin{array}{c}\vdots\ \delta_2\\ C \supset D\end{array}}{B \supset D}\,{\supset}\mathrm{E}$$

- p. 132, section 4.6, 1st para, l. 5: “For instance, instead of” → “For instance, you can replace”; “deduce” → “by” in the following display; delete “instead.” after display

- p. 133, 1st full para after Definition 4.26, l. −3: move closing parenthesis to end of sentence.

- p. 134, proof of proposition 4.28, l. 1: "cuts" → "cut"
- p. 134, para below the proof of Proposition 4.28, l. −4: "$A \wedge B_b$" → "$A \wedge C_b$"
- p. 136, l. 12: "discharged, (See Section 3.2.3 for an example.)" → "discharged (see Section 3.2.3 for an example)."
- p. 136, after first display: replace "premise" by "disjunct"
- p. 138, first displayed deduction: "C" → "D"
- p. 147, l. 1: "b, d, f, h, and j" → "$A \supset C_b$, $A \supset C_d$, C_f, B_i, and A_j"
- p. 148, Proposition 4.36, (3): "$m < i < k$" → "$m < i < \ell$"
- p. 151, Definition 4.39, (1): "Base" → "Basis"
- p. 152, proof of Lemma 4.41, l. −2: "But all formulas between A_k and the end-formula of δ lie on paths of order $< o$." → "But all formulas occuring in δ between A_k and the end-formula lie on paths of order $< o$."
- p. 153, l. 7: "of a path" → "of a formula on a path"
- p. 154, Corollary 4.47 should read: "**NJ** does not prove all instances of $A \vee \neg A$."
- p. 154, Corollary 4.49 should read: "**NM** does not prove all instances of $(A \wedge \neg A) \supset B$."
- p. 159, l. 1: "cut segment" → "segment"
- p. 159, l. 5: "any proof" → "any proof δ"
- p. 159, l. −4: "subdeduction" → "sub-deduction"
- p. 160, l. 1: "$s(\delta) = 0$" → "$s(\delta) = 1$ and $s(\delta^*) = 0$"
- p. 160, l. 3: "$r(\delta^*) = 1$" → "$r(\delta) = 1$"
- p. 167, l. 8: "is an I-rule" → "is the conclusion of an I-rule"
- p. 167, l. −1: Add to the end of the last paragraph: "We leave the verification of this as an exercise."

Chapter 5

- Throughout chapter 5 and 6: in inductive definitions, we number inductive clauses together with basis and extremal clause. For uniformity with other chapters, introduce subsidiary numbering for the inductive clauses. (Definition 5.4, Theorem 5.28, Theorem 5.31, Definition 6.29)
- p. 168, 1st para, l. 3: "the notion of *sequents*" → "the notion of *sequent*"
- p. 168, 1st para, l. 9: "number" → "finite number"
- p. 169, replace "The intuitive meaning of a sequent . . . " to the end of the page with:

  Assuming we have a semantics that makes formulas true or false, we can give an intuitive interpretation of when a sequent holds or doesn't: $\Gamma \Rightarrow \Delta$ holds if, and only if, one of the formulas A_i in Γ is false, or one of the formulas B_j in Δ is true. In other words, it holds if, and only if, whenever all the formulas in Γ are true, then at least one of the formulas in Δ is true. If Γ is empty, it holds if, and only if, at least one of the B_j is true; if Δ is empty, when at least one of the A_i is false. Consequently, the empty sequent $\Rightarrow$ never holds.

  We can translate a sequent $A_1, \dots, A_n \Rightarrow B_1, \dots, B_n$ into a single formula that is true if, and only if, the sequent holds as follows (this is the translation we would use to show that the sequent calculus is equivalent to axiomatic derivations, as we did for natural deduction in Section 3.7):

  1. $n \neq 0, m \neq 0$: $\Gamma \Rightarrow \Delta$ translates to
  $$(A_1 \wedge \cdots \wedge A_n) \supset (B_1 \vee \cdots \vee B_m).$$
  2. $n = 0, m \neq 0$: Γ is empty, and $\Rightarrow \Delta$ translates to
  $$B_1 \vee \cdots \vee B_m$$
  3. $n \neq 0, m = 0$: Δ is empty, and $\Gamma \Rightarrow$ translates to
  $$\neg(A_1 \wedge \cdots \wedge A_n).$$
  4. $n = 0, m = 0$: $\Rightarrow$, i.e., the empty sequent, translates to the contradiction $\perp$ (see Section 2.10.1).

- p. 170: Replace 2nd full para

  As will become clear after we present the axioms and rules, each line of a derivation must be valid and thus sequent systems are more like axiomatic ones than natural deduction systems.

by

> Proofs in **LK** are more like axiomatic derivations (considered as trees as in Section 2.7) than they are like deductions in **NK**. The initial sequents (axioms) of **LK**-proofs are like the axioms of derivations. Axiom sequents of the form $A \Rightarrow A$ always hold on their intuitive interpretation, just like the axioms in axiomatic derivations are always true: since A is either true, in which case a formula in the succedent is true, or it is false, in which case a formula in the antecedent is false. Unlike assumptions in natural deduction, they do not need to be discharged in order to prove a sequent unconditionally.

- p. 171, section 5.2.4, 1st para, l. 2: "logical operators and quantifiers" $\rightarrow$ "logical operators (connectives and quantifiers)"
- p. 173. Delete numbers from paragraphs labelled (1) and (2). Delete the entire paragraph labelled (3). Add at end of paragraph labelled (1): "(They are sometimes also called *context formulas*.)"
- p. 175, Definition 5.7, (2), l. 2: "these" $\rightarrow$ "its"
- p. 180, l. 7: "a deduction" –> "an **NK**-deduction"
- p. 180, Problem 5.8, l. 1: "structure" $\rightarrow$ "structure of"
- p. 184, para after the proof of (a): "critical conditions for quantifiers rules" $\rightarrow$ "eigenvariable conditions of $\exists$L and $\forall$R inferences"
- p. 185, l. 1 after first proof: "therefore d " $\rightarrow$ "therefore sequent (d)"
- p. 187, Section 5.6, 1st para, l. 1: "logical initial sequents" $\rightarrow$ "logical initial sequents, i.e., initial sequents of the form $A \Rightarrow A$,"
- p. 188, Section 5.7, 1st para, l. 3: "Γ' results from Γ" $\rightarrow$ "Γ' results from Γ, and Δ' from Δ,"
- p. 189, l. 2: "inferences" $\rightarrow$ "proofs"
- p. 190, l. 1 after first display: 'eigenvaraible" $\rightarrow$ "eigenvariable"
- p. 192, l. 5: "deductions" $\rightarrow$ "proofs", "proofs" $\rightarrow$ "deductions"
- p. 194, last display: "$\Rightarrow$" $\rightarrow$ "or"
- p. 197, 1st para, l. 2: "$\Gamma \Rightarrow E$" $\rightarrow$ "$\Gamma \Rightarrow A$"
- p. 197, Proof of Theorem 5.31, l. 1: Replace

> We define a mapping of regular **LJ**-proofs π with end-sequent $\Gamma \Rightarrow [A]$ to **NJ**-deductions $D(\pi)$ of A (or $\perp$, if the succedent of the end-sequent is empty) with $\mathrm{As}(D(\pi)) \subseteq \Gamma$ by induction.

by

> We define a mapping D of regular **LJ**-proofs π to **NJ**-deductions $D(\pi)$ by induction. We'll use $\Gamma \Rightarrow [A]$ to indicate a sequent where the succedent may be empty or else contains the single formula A. If $\Gamma \Rightarrow [A]$ is the end-sequent of π, then $D(\pi)$ is a deduction of A (or of $\perp$ if the succedent of the end-sequent is empty) with $\mathrm{As}(D(\pi)) \subseteq \Gamma$.

- p. 198, paragraph following the second display, l. 3: "[A] indicates that A may occur on the right side of the end-sequent of π, or the right side of the sequent may be empty." → "Recall that [A] indicates that A may occur as the only formula in the succedent of the end-sequent of π, or the succedent of the end-sequent may be empty."

Chapter 6

- p. 203, Theorem 6.3, l. 2: "proof of" → "proof in"
- p. 203, 2nd para after Theorem 6.3, l. 2: "if two" → "if several"
- p. 204, l. 2: "by transforming" → "by showing how to transform"
- p. 204, Definition 6.7, l. 2: "path" → "thread"; l. 4: Replace

> A path is understood as a sequence of sequents in which the successor relation coincides with the relation of being one of the premises of the immediately following sequent.

by

> A thread is a sequence of occurrences of sequents in the proof where each sequent is a premise of an inference, the conclusion of which is the following sequent in the thread.

- p. 204, Definition 6.7, 2nd para, l. 2: "path" → "thread"
- p. 207, 2nd para of inductive step, l. −3: "side-formulas" → "auxiliary formulas"
- p. 208, 3 lines below second display: "path" → "thread"
- p. 208, 7 lines below second display: "end-formula" → "end-sequent"
- p. 212, l. −4: "leading to he right" → "leading to the right"

- p. 214, case D3: replace "Γ" by "Γ_2"
- p. 220–231: "Ω_1" → "Θ"
- p. 221, **B2. Rule** IR: insert: "See Problem 6.20"
- p. 223, para below first display, l. 1: "to that of the" → "to the"; l. −2: "cut-free derivation" → "derivation without MIX"
- p. 227, l. 1: "In this case" → "In this case, as long as $M \neq F(t)$,"
- p. 230, case B10, l. 1 after the first display: "It can be" → "If $M \neq A$, the proof can be"
- p. 244, l. −1: "Section 6.7" → "Figure 6.1"
- p. 247, 1st para before 1st display, l. 3: "Case (A)" → "case A"
- p. 248, l. 1: "Figure 6.7" → "Figure 6.1"
- p. 250, Problem 6.23, "proofs c and d" → "proofs (c) and (d)"
- p. 254, Definition 6.29: Number inductive cases (a)–(c).
- p. 254, section 6.10, 1st para, l. 3: "one cannot have access to it" → "one cannot appreciate this significance"
- p. 256, Proposition 6.33 should read: "**LJ** does not prove all instances of $\Rightarrow A \vee \neg A$."
- p. 256, replace the proof of Proposition 6.33 with:

  > Suppose it did. Then it would have to prove, in particular, all instances of $\Rightarrow A \vee \neg A$ for A *atomic*. Then by Theorem 6.32, **LJ** proves either $\Rightarrow A$ or $\Rightarrow \neg A$. If it proves the latter, it must also prove $A \Rightarrow$, since the only possible last inference of a cut-free proof of $\Rightarrow \neg A$ in **LJ** is ¬R. Consequently, **LJ** proves either $\Rightarrow A$ or $\Rightarrow \neg A$. However, a cut-free proof of $\Rightarrow A$ or of $\Rightarrow \neg A$, where A is atomic, cannot contain any operational inferences, by Theorem 6.30 (subformula property). On the other hand, weakening, contraction, and interchange rules cannot produce a sequent with an empty succedent or antecedent unless the premise already had an empty succedent or antecedent, respectively. Thus, there can be no cut-free proof of either $\Rightarrow A$ or $A \Rightarrow$ at all, for atomic A.

- p. 256, Corollary 6.34 should read: "**LM** does not prove all instances of $\Rightarrow (A \wedge \neg A) \supset B$."
- p. 256, proof of Corollary 6.34 should read:

It is an easy exercise to find a proof in **LJ** of

$$\neg\neg(A \lor \neg A) \supset (A \lor \neg A) \Rightarrow A \lor \neg A.$$

If there were derivations of all instances of $\Rightarrow \neg\neg A \supset A$, there would be one of $\Rightarrow \neg\neg(A \lor \neg A) \supset (A \lor \neg A)$ in particular. Using a CUT, we would obtain a proof of $\Rightarrow A \lor \neg A$, contradicting Proposition 6.33.

- p. 256, Corollary 6.35 should read: "**LM** does not prove all instances of $\Rightarrow (A \land \neg A) \supset B$"
- p. 259, l. 2–3: delete "Since the axioms contain no quantifiers, the premise of I contains no quantifiers."
- p. 259, 2nd full para, l. 2: "*and*" → "*and* in which"
- p. 259, 4th para, l. 1: delete "Inductive step:"
- p. 260, l. 1 after second display: "by adding" → "by applying"
- p. 261, first full para, l. 1: Italicize "Inductive step:"
- p. 268, l. −5: "form $B(t_1, \ldots, t_n)$" → "form $\Rightarrow B(t_1, \ldots, t_n)$"

Chapter 7

- p. 269, 2nd para, l. −1: "Thus" → "Thus $\overline{0}$ abbreviates 0,"
- p. 271, fn. 2, l. 2: "axiom sequents" → "initial sequents"
- p. 275, l. 1 after Problem 7.8: "Any proof in **LK** using the induction rule CJ and making use of mathematical initial sequents counts as a proof in **PA**." → "A proof in **PA** is a proof using the rules of **LK** or the induction rule CJ, and in which all initial sequents are either logical initial sequents or mathematical initial sequents, i.e., a instances of PA_S1–PA_S10."
- p. 275, l. −3: "**PA**" → "**LK**"
- p. 279, 1st full para after Problem 7.11, l. 4: "$\neg(0 = 0) \Rightarrow$ " → "$\neg\, 0 = 0 \Rightarrow$ "
- p. 280, l. 1 after second display: " $\Rightarrow s = u$" → " $\Rightarrow t = u$"
- p. 281, Proposition 7.12, Induction basis: "the base case" → "the induction basis"
- p. 281, l. −2: "Here's the" → "Here's a'
- p. 282, Proposition 7.16: "proof" → "proof in **PA**"

- p. 283, l. 1 of Proposition 7.19 and 7.20: "regular proof" → "regular proof in **PA**"
- p. 285, 5th para, l. 3: "on that formula" → "on that formula (if any)"
- p. 286, fn. 9: Insert at beginning: "Recall that"
- p. 286, last para, l. 4: "lowermost implicit" → "These lowermost implicit operational"
- p. 287, add to end of last para of Section 7.3: "In particular, the end-part itself contains no operational inferences at all. "
- p. 289, Proposition 7.27: "Suppose π is a regular proof without free variables other than eigenvariables." → "Suppose π is a regular proof in **PA** of an atomic sequent in which all free variables are eigenvariables."
- p. 289, proof of Proposition 7.28, l. 3: "closed term $\overline{n}$." → "closed term $\overline{n}$. (Recall for simplicity we are, for now, not allowing + and · in our terms.)"
- p. 291, 2nd full para, l. −3: "So, in the new proof, the maximal length of induction chains is $\leq m$, and the number of induction chains of length m has been decreased by at least 1." → "We have turned at least on induction chain of length m into (possibly many) induction chains of length $< m$. If π^* is the resulting proof, then either $m(\pi^*) < m$ (if the induction chain considered was the only one of length m) or $m(\pi^*) = m$ and $o(\pi^*) < o(\pi)$."
- p. 292, 4th displayed proof, right premise of the last cut: "$F(0'') \Rightarrow F(0'')$" → "$F(0') \Rightarrow F(0'')$"
- p. 294, l. 2 below the first display: "contains the formula" → "also contains the formula"
- p. 294, l. 3 below the first display: Add a footnote: "You might wonder why we have to use a cut on $\forall x\, F(x)$ and can't just replace the original cut on $F(\overline{n})$ alone. The reason is that π_1' and π_2' may contain $\forall x\, F(x)$ on the right or left, respectively, which are contracted. The cut on $\forall x\, F(x)$, deals with all these formulas, not just those arising from the ∀r and ∀l inferences displayed."
- p. 294, 1st full para, l. 2: Replace

  > Both operational boundary inferences are still present (at the end of the left copy of π_2 and at the end of $\pi_1(a)$), plus a copy each of the proofs ending in their premises with the operational inference replaced by a weakening ($\pi_1(\overline{n})$ and the right copy of π_2).

  by

> The sub-proofs containing the operational boundary inferences are still present (leading to the right premise of the left CUT on $\forall x\, F(x)$ and the left premise of the right CUT.). We have added copies of these sub-proofs, with operational inference replaced by a weakening (these are the sub-proofs above the left and right premise of the upper two CUTS).

- p. 296, l. 5 below the first display: "choose on" → "choose one"
- p. 297, l. −2 before the last display: "Including displaying" → "If we also display"
- p. 299, l. 2 after 3rd display: Replace

  > Again, the inductive hypothesis yields a proof π_1^* with the end-sequent being $\Gamma^*, A, \Delta^* \Rightarrow \Theta^*$, or $\Gamma^*, B, \Delta^* \Rightarrow \Theta^*$, or $\Gamma^*, A, B, \Delta^* \Rightarrow \Theta^*$. In the first two cases, let $\pi^* = \pi_1^*$; in the third, π^* is

  by

  > Again, the inductive hypothesis yields a proof π_1^* with the end-sequent being one of:
  >
  > $$\begin{array}{rr} \Gamma^*, \Delta^* \Rightarrow \Theta^* & \Gamma^*, A, \Delta^* \Rightarrow \Theta^* \\ \Gamma^*, B, \Delta^* \Rightarrow \Theta^* & \Gamma^*, A, B, \Delta^* \Rightarrow \Theta^* \end{array}$$
  >
  > In the first three cases, let $\pi^* = \pi_1^*$; in the last, π^* is

- p. 299, l. −5: "sub-sequents of" → "sub-sequents"
- p. 302, l. 2 below the first display: "on" → "at"
- p. 303, 1st para, l. −2: "starts with" → "contains"
- p. 303, l. 4 after the display: "of principal" → "of the principal"
- p. 305, l. 3: "It also" → "The end-part also"
- p. 305, l. 6–7: delete "How do we remove such a suitable CUT?"
- p. 305, 1st full para, l. −6: "In the" → "How do we remove such a suitable CUT? In the"
- p. 307, last para, l. −3: "conclusion a" → "conclusion of a"
- p. 307, fn. 13: "In the more general case we're proving here, there" → "There"

Chapter 8

- p. 313, Definition 8.3, add at end: "If $<$ is a well-ordering of X we say that X is well-ordered by $<$ and that $\langle X, <\rangle$ is a well-ordered set."
- p. 314, Proof of Proposition 8.4, last para, l. 1: "no u" → "there is no u"
- p. 314, Problem 8.7: "total ordering" → "strict linear ordering"
- p. 316, para following Problem 8.12, l. 2–4: "digit" → "number"; l. 3: "(word)" → "(words)"
- p. 316, l. −1 above Proposition 8.14: "total" → "strict linear"
- p. 317, Problem 8.15, last line of display: "or or" → "or"
- p. 319, l. −1 before Proposition 8.18: "well-ordered by $\prec_{\text{lex}}$." →

  well-ordered by $\prec_{\text{lex}}$, that is,

  $$\langle \underbrace{x, \ldots, x}_{n\ x\text{'s}} \rangle \prec_{\text{lex}} \langle \underbrace{y, \ldots, y}_{m\ y\text{'s}} \rangle \text{ if, and only if, either } x < y \text{ or } x = y \text{ and } n < m.$$

- p. 319, l. −1: "Exercise" → "Left as an exercise"
- p. 320, proof of Proposition 8.20. Replace

  $(X^*_=)^*_{\succ_{\text{lex}}}$ is ordered by the lexicographical order $\ll$ based on $\prec_{\text{lex}}$ itself, i.e., the ordering of sequences of constant sequences of elements of X where

  $$\langle \underbrace{x, \ldots, x}_{n\ x\text{'s}} \rangle \ll \langle \underbrace{y, \ldots, y}_{m\ y\text{'s}} \rangle \text{ if, and only if, either } x < y \text{ or } x = y \text{ and } n < m.$$

  by

  $(X^*_=)^*_{\succ_{\text{lex}}}$ is ordered by the lexicographical order $\ll$ based on $\prec_{\text{lex}}$ itself, i.e., the ordering of sequences of constant sequences of elements of X. That is, if

  $$s = \langle y_1, \ldots, y_k \rangle$$
  $$t = \langle y'_1, \ldots, y'_\ell \rangle$$

  where the y_i, y'_i are constant sequences, then $s \ll t$ holds if, and only if, one of the following conditions is satisfied:

  1. $k < \ell$ and $y_1 = y'_1, \ldots, y_k = y'_k$, or

2. there is a $j < k$ such that $y_1 = y'_1, \ldots, y_j = y'_j$ and $y_{j+1} <_{\text{lex}} y'_{j+1}$, where $<_{\text{lex}}$ is the lexicographic ordering of constant sequences.

- p. 320, Section 8.3, 1st para, l. 4: “Cantor-Bachmann” → “Cantor (or Cantor-Bachmann)”
- p. 321, 3rd full para, l. 3: “non-decreasing” → “non-increasing”
- p. 321, 4th full para, l −4: “all powers” → “all sums of powers ”
- p. 321, 4th full para, l. −1 and −2: “ordinals” → “ordinal notations” (twice)
- p. 321, l. −3: “ordinals” → “ordinal notations”
- p. 322, l. 3 after the second display: “followed” → “is followed”
- p. 322, l. −2: “α and $\beta \in O_{\leq k}$” → “$\alpha, \beta \in O_{\leq k}$”
- p. 324, proof of Proposition 8.25, l. 4: “clause c” → “clause (c) of Definition 8.21”; 2nd para, l. 3: “clause a” → “clause (a)”; l. 3 and l. 6: “clause b” → “clause (b)”
- p. 325, l. 2: “clause b” → “clause (b)”; l. 2 and 3: “clause c” → “clause (c)”
- p. 325, fn. 2, l. 2: “such that” → “such that (b)”
- p. 325, fn. 3: “$\alpha_i \prec \beta_i$ or $\beta_j \prec \alpha_i$ or $\alpha_i = \beta_i$” → “$\alpha_i \prec \beta_j$ or $\beta_j \prec \alpha_i$ or $\alpha_i = \beta_j$”
- p. 329, Proposition 8.38: add full stop at end
- p. 330, l. −3: “and the least” → “and for the least”
- p. 331, l. 4: ““clause b or clause c” → “clause (b) or clause (c) of Definition 8.21”; l. 5: “clause b” → “clause (b)”; l. 8: “clause c” → “clause (c)”
- p. 331, 1st para after Problem 8.44, l. 3: “from the previous section” → “from Section 8.2”
- p. 331, 1st para after Problem 8.44, l. −2: “$\alpha_1 \geq_k \alpha_{i+1}$” to $\alpha_i \succeq_k \alpha_{i+1}$
- p. 331, Proposition 8.45 and Corollary 8.46: “well-ordering” → “well-ordered set”
- p. 331, proof of Corollary 8.46: “ordinal” → “ordinal notation” (3 times)
- p. 332, l. 7–8: “non-decreasing” → “non-increasing”

- p. 333, Definition 8.48 should read:

  A set α is an *ordinal* if, and only if,

  1. it is transitive, i.e., for all $\beta \in \alpha$, $\beta \subseteq \alpha$, and
  2. $\langle \alpha, \in \rangle$ is a well-ordering.

- p. 335, proof of Proposition 8.56: "by the previous problem" $\to$ "by Problem 8.51"

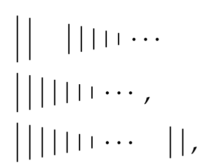
- p. 335, proof of Proposition 8.57, l. 2–4: Replace

  Just as in the case of ω discussed above, any such set is itself an ordinal, since every member of it is an ordinal, and an ordinal is equal to the set of ordinals less than it. If X is downward closed, all these ordinals are members of X.

  by

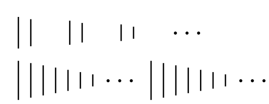
  Just as in the case of ω discussed above, any downward closed set X is itself an ordinal: every member of it is an ordinal, an ordinal is equal to the set of ordinals less than it, and since X is downward closed, all these ordinals are members of X.

- p. 337: Replace matchstick diagrams by:



- p. 338, l. 3: replace matchstick diagram by " || "

- p. 338: Replace matchstick diagrams by:



- p. 338, Replace 4th display

  $$\langle k_1, \dots, k_n \rangle \leq \langle k'_1, \dots, k'_n \rangle \text{ if, and only if, } k_j < k'_j \text{ and, for some } j \leq n, \\ k_i = k'_i \text{ for } i = j+1, \dots, n.$$

  by

  $$\langle k_1, \dots, k_n \rangle \leq \langle k'_1, \dots, k'_n \rangle \text{ if, and only if, for some } j \leq n, \\ k_j < k'_j \text{ and } k_i = k'_i \text{ for } i = j+1, \dots, n.$$

- p. 339, l. 3: “is ε_0” → “. . . is ε_0”
- p. 341, 2nd para after Problem 8.68, l. 1: “in the previous section” → “above”
- p. 342, l. −2: “ordinals” → “ordinal notations”
- p. 343, l. 1: “ordinal” → “ordinal notation”
- p. 343, l. 1 after first display: “the ordinals β_i are the ordinals” → “the β_i are the ordinal notations”
- p. 343, 1st full para, l. 1: “immediate predecessor” → “parent”
-
- p. 343, 1st full para, l. 2: “ordinal” → “ordinal notation”
- p. 343, 1st full para, l. 3: “descendant” → “parent’s parent”
- p. 343, l. 1 after 2nd display: “ordinal” → “ordinal notation”
- p. 343, Problem 8.70 should read:

  > Compute the ordinal notations assigned to nodes in the tree resulting by removing the head in the previous example. Verify that the ordinal notation assigned to the root is less than that assigned to the root of the original tree, $\boldsymbol{\omega}^{\omega^{\omega^{0}\cdot 3}+\omega^{0}\cdot 2} + \boldsymbol{\omega}^{\omega^{0}\cdot 2} + \boldsymbol{\omega}^{\mathbf{0}}$.

- p. 345, l. 3: “ordinal” → “ordinal notation”

Chapter 9

- p. 347, last para, l. 1, 2, 4, 5: “path” → “thread” (4 times)
- p. 349, Definition 9.3, inductive clause: introduce subsidiary numbering for the inductive clauses.
- p. 348, 1st full para, l. −4, −1: “path” → “thread” (2 times)
- p. 348, 3rd full para, l. 3: “mix” → “MIX”
- p. 350, l. −3 before Problem 9.4: ‘conclusion’ → ‘“conclusion”’
- p. 356, 1st full para, l. 4: “their” → “its”
- p. 356, 3rd full para, l. 3: “the last of” → “any lowermost CUT among”
- p. 356, 3rd full para, l. 4: “than” → “than the one assigned to”

- p. 358, l. 3 below the last display: "essenial" → "complex"; "own-end-part" → "own end-part"
- p. 359, l. 1 below the 1st display: "The example also illustrates how the ordinal notation assigned to CJ inferences increases." → "This example also illustrates how the ordinal notation increases when going from the premise to the conclusion of a CJ inference."
- p. 360, Section 9.3, l. 4 of 1st para: "$\Gamma^* \Rightarrow \theta^*$" → "$\Gamma^* \Rightarrow \Theta^*$"
- p. 361, l. 2: "inferences" → "inference"
- p. 361, l. −2 above display: "it is cut-formula of" → "of which B is the cut-formula"
- p. 362, 1st full para, l. −2: "never greater" → "never strictly smaller"
- p. 362, Proposition 9.12, (1): add period at end
- p. 362, Proposition 9.12, (5), l. 2: "sequents in π" → "sequents"
- p. 363, proof of Proposition 9.12, Inductive step, 1st para: add at end: "(In this case, the end-part consists only of the end-sequent, and so the proof is not simplified.)"
- p. 363, l. 1 after 2nd display: "$\alpha = o_\ell(A, \Gamma \Rightarrow \theta; \pi)$" → "$\alpha = o_\ell(A, \Gamma \Rightarrow \Theta; \pi)$"
- p. 365, 2nd and 3rd displa: add period at end
- p. 367, last para, l. 2: "on these labels" → "on the labels inherited from π."
- p. 367, l. −3: "levels by" → "level labels by"; "decreasing levels" → "decreasing labels"
- p. 367, l. −2: "levels" → "labels"
- p. 368, l. 1 after the second display: "readjusting the levels to their correct values" → "readjusting the labels to match the actual levels of the corresponding sequents"
- p. 368, Section 9.4, l. 3: "premise" → "premises"
- p. 372, last para, l. 3: "other complex CUTS" → "one or more complex CUTS"
- p. 374, l. −4: "the same as" → "the same as those of"
- p. 374, l. −3: "corresponding levels of sequents" → "levels of the corresponding sequents"
- p. 375, add periods at the ends of the last two displays

Appendices

- p. 381, Definition B.1, l. 5: "property and object x might have" → "property"
- p. 381, Definition B.4, l. 1: "intersection" → "*intersection*"
- p. 381, Definition B.4, 2nd displayed equation should read:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

- p. 384, section C.2, l. 1: "the theorem is provable in $\mathbf{M}_0$" → "a theorem is provable in $\mathbf{M}_0$ or $\mathbf{M}_1$"
- p. 384, 2nd para: delete subscript "$\mathbf{M}_0$" from the statements of the derived rules.
- p. 385: delete subscripts "$\mathbf{M}_1$" and "$\mathbf{S}_1$"
- p. 386, 2nd full para, l. 1: "Start with axioms" → "Start with axiom"
- p. 394, insert after the table of rules:

  > The rule ∀I and ∃E are subject to eigenvariable conditions. In ∀I, the variable c must not occur in any open assumption. In ∃E, the variable c must not occur in C, nor in any assumption open in the sub-deduction leading to the minor premise C, except the assumptions of the form $A(c)$, all of which must be discharged at the inference.

- p. 395, E.2.1: "conversion" → "conversions"
- Appendix G: change all occurrences of "deg" to "dg" and all occurrences of "rk" to "rk"